On the numerical implementation of a fully coupled gravity-wave ray tracer in atmospheric models

Gergely Bölöni, Bruno Ribstein, Ulrich Achatz, Jewgenija Muraschko, Christine Sgoff, Junhong Wei
GWs are not fully resolved by GCMs and NWP models ⇒ parametrization ⇒ (Wentzel–Kramers–Brillouin) WKB theory

Currently used parametrizations: equilibrium profile of the wave field, steady state background wind, fully linear theory ⇒ the resolved flow (mean flow) does not know about the GWs propagating through it ⇒ the GW momentum is deposited instantenously to the mean flow at height of wave breaking

Proposal for improvement: weakly nonlinear coupling between the waves and the resolved flow ⇒ continuous interaction between the wave and the mean flow during the propagation
Wave resolving equations (2-D Euler equations, no rotation):

\[
\begin{align*}
\frac{Du}{Dt} + c_p \theta \frac{\partial \pi}{\partial x} &= 0 \\
\frac{Dw}{Dt} + c_p \theta \frac{\partial \pi}{\partial z} + g &= 0 \\
\frac{D\theta}{Dt} &= 0 \\
\frac{D\pi}{Dt} + \frac{\kappa}{1 - \kappa} \pi \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) &= 0
\end{align*}
\]

with \( \frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \)

Exner pressure \( \pi = (p/p_0)^\kappa \)

Pot. temperature \( \theta = T(p_0/p)^\kappa = T/\pi \)

\( \kappa = R/c_p \)
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Exner pressure
Pot. temperature

Simplification ingredients:

- Decomposition of the fields: \( f = \bar{f} + f_b + f_w \)
WKB theory

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- Decomposition of the fields: \( f = \bar{f} + f_b + f_w \)
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  with \( Z = \epsilon z, T = \epsilon t, m = \partial \phi / \partial Z \) and \( \omega = -\partial \phi / \partial T \)
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  with \( Z = \epsilon z, T = \epsilon t, m = \partial \phi / \partial Z \) and \( \omega = -\partial \phi / \partial T \)
- Scaling for the gravity waves: \( \epsilon = L_w / H_\theta \ll 1 \): weak stratification
WKB theory

- At leading order $O(\epsilon^2)$: dispersion-, and polarization relations $\Rightarrow$ ray equations

Problem: if rays crossing $\Rightarrow$ caustics: several at same height $\Rightarrow$ e.g. $c_g z$ multivalued BUT!

$A = A(z,t) \Rightarrow$ wave action conservation ill-defined $\Rightarrow$ numerical problems
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- At next order $O(\epsilon^3)$: wave action conservation and the mean-flow equations
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- The coupled system (Achatz et al., 2010, Muraschko et al., 2015):

**Wave field**

\[
\begin{align*}
\frac{dg_z}{dt} &= \mp \frac{Nkm}{(k^2 + m^2)^{3/2}} \equiv c_{gz} \\
\frac{dg_m}{dt} &= \mp \frac{k}{(k^2 + m^2)^{1/2}} \frac{dN}{dz} - k \frac{du_b}{dz} \equiv \bar{m} \\
\frac{dA}{dt} &= -A \frac{\partial c_{gz}}{\partial z} \left( \frac{dg}{dt} = \frac{\partial}{\partial t} + c_{gz} \frac{\partial}{\partial z} \right) 
\end{align*}
\]

**Mean flow**

\[
\begin{align*}
\frac{\partial u_b}{\partial t} &= -\frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \left[ \frac{\bar{\rho}}{2} \text{Re} (U_w W_w^*) \right] \\
\frac{d\bar{m}}{dt} &= -\frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (k c_{gz} A) \\
\bar{\rho} &= \rho(z) = \rho_0 e^{-z/H} \quad H = \frac{RT_0}{g}
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  \frac{\partial}{\partial t} \left[ \rho \right] = -\frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (k c_{gz} A)
  \]

  - $\bar{\rho}$

  \[
  \bar{\rho} = \rho(z) = \rho_0 e^{-z/H} \quad H = \frac{RT_0}{g}
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- Problem: if rays crossing $\Rightarrow$ caustics: several $m$ at same height $z$ $\Rightarrow$ e.g.
  $c_{gz}$ multivalued BUT! $A = A(z, t) \Rightarrow$ wave action conservation ill-defined
  $\Rightarrow$ numerical problems
**WKB theory in phase space**

- Solution: extension of the model to a 2D phase space $(z, m)$

![Diagram showing $N(z, m, t)$ in a 2D phase space with axes $z$ and $m$.]

_Hertzog et al., 2002,
Muraschko et al., 2015_
WKB theory in phase space

- Solution: extension of the model to a 2D phase space \((z, m)\)
- "Slicing up" the wave action density to several \(m\) intervals \(\Rightarrow\) phase-space wave action density:

\[
N(z, m, t) = \int_{R} A_\alpha(z, t) \delta[m - m_\alpha(z, t)] d\alpha
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\[Hertzog \text{ et al., 2002, Muraschko et al., 2015}\]
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- Eulerian view

\[
\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial (c_g z \mathcal{N})}{\partial z} + \frac{\partial (m \mathcal{N})}{\partial m} = 0
\]

Hertzog et al., 2002,
Muraschko et al., 2015
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\]

Eulerian view

\[
\frac{\partial N}{\partial t} + \frac{\partial (c_g z N)}{\partial z} + \frac{\partial (\dot{m} N)}{\partial m} = 0
\]

... and we have

\[
\frac{\partial c_g z}{\partial z} + \frac{\partial \dot{m}}{\partial m} = 0
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\frac{\partial \mathcal{N}}{\partial t} + \frac{\partial (c_{gz} \mathcal{N})}{\partial z} + \frac{\partial (\dot{m} \mathcal{N})}{\partial m} = 0
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- Lagrangian view

\[
\frac{\partial \mathcal{N}(z, m, t)}{\partial t} + c_{gz} \frac{\partial \mathcal{N}(z, m, t)}{\partial z} + \dot{m} \frac{\partial \mathcal{N}(z, m, t)}{\partial m} = 0
\]

Hertzog et al., 2002, Murashko et al., 2015

Coupled gravity-wave raytracing
WKB theory in phase space

- Coupled wave - meanflow equations in phase space:

**Wave field**
\[
\frac{dr}{dt} z = \mp \frac{N km}{(k^2 + m^2)^{3/2}} \equiv c_{gz} \\
\frac{dr}{dt} m = \mp \frac{k}{(k^2 + m^2)^{1/2}} dN - k \frac{d u_b}{dz} \equiv \dot{m} \\
\frac{dr}{dt} \mathcal{N} = 0 \left( \frac{dr}{dt} = \frac{\partial}{\partial t} + c_{gz} \frac{\partial}{\partial z} + \dot{m} \frac{\partial}{\partial m} \right)
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**Mean flow**
\[
\frac{\partial u_b}{\partial t} = -\frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \left[ \bar{\rho} \text{Re}(U_w W^*) \right] \\
\frac{\partial u_b}{\partial t} = -\frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (k c_{gz} \mathcal{A}) \\
\frac{dr}{dt} \mathcal{N} = -\frac{1}{\bar{\rho}} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} k c_{gz} \mathcal{N}(z, m, t) dm
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WKB theory in phase space

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\begin{align*}
\frac{dr_z}{dt} &= \mp \frac{N km}{(k^2 + m^2)^{3/2}} \equiv c_{gz} \\
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\frac{d_r N}{dt} &= 0 \left( \frac{dr}{dt} = \frac{\partial}{\partial t} + c_{gz} \frac{\partial}{\partial z} + \dot{m} \frac{\partial}{\partial m} \right)
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\frac{\partial u_b}{\partial t} &= -\frac{1}{\rho} \frac{\partial}{\partial z} \left[ \frac{\rho}{2} \Re (U_w W^*) \right] \\
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\frac{d_r N}{dt} &= -\frac{1}{\rho} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} k c_{gz} N(z, m, t) dm
\end{align*}
\]

- phase space wave action density \( N \) conserved along ray trajectories
WKB theory in phase space

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\frac{d_r N}{dt} = 0 \left( \frac{d_r}{dt} = \frac{\partial}{\partial t} + c_{gz} \frac{\partial}{\partial z} + \dot{m} \frac{\partial}{\partial m} \right) = -\frac{1}{\rho} \frac{\partial}{\partial z} \int_{-\infty}^{\infty} k_{cgz} N(z, m, t) dm
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- multiple $m$ values allowed at each location $z$ \(\rightarrow\) spectral (non-monochromatic) treatment \(\rightarrow\) no caustics problems
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- phase space wave action density \( \mathcal{N} \) conserved along ray trajectories

- multiple \( m \) values allowed at each location \( z \rightarrow \) spectral (non-monochromatic) treatment \( \rightarrow \) no caustics problems

- extension to Muraschko et al., 2015: isothermal background with variable density \( \bar{\rho} = \rho(z) = \rho_0 e^{-z/H} \)

\[ H = \frac{RT_0}{g} \]
Wave breaking parametrization

- Weakly nonlinear WKB: wave breaking not considered
Wave breaking parametrization

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- Saturation occurs if static instability sets in (Lindzen 1981), i.e.
  \[ \frac{\partial \theta_w}{\partial z} + \frac{d\theta}{dz} < 0 \]
  or, after an additional multiplication by \( g/\bar{\theta} \)
  \[ \frac{\partial b_w}{\partial z} + N^2 < 0 \]
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- with the ansatz \( b_w(x, z, t) = \text{Re} B_w(Z, T) e^{i \left[ kx + \phi(Z, T) \right] / \epsilon} \) with \( m = \partial \phi / \partial Z \) this amounts in
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  |m| |B_w| > N^2
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  with \( m = \partial \phi / \partial Z \) this amounts in
  \[ |m||B_w| > N^2 \]

- This monochromatic criterion can be generalized for a spectrum by taking its integral over the vertical wavenumber:
  \[ \int_{-\infty}^{\infty} (m|B_w|)^2 dm = \int_{-\infty}^{\infty} m^2 f(N) dm > \alpha^2 N^4 \]
  where \( \alpha \) is a parameter accounting for the uncertainty of the criterion
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- If the saturation criterion is fulfilled the wave action density \( N \) is reset layer-wise to a value that sets back stability
Numerical experiments

Methodology

- LES: fully non-linear wave resolving reference (PincFloit, Rieper et al., 2013)
- WKB-eu: Eulerian WKB model
- WKB-la: Lagrangian WKB model
Numerical experiments

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Cases

- refraction by a jet: $U_{jet}$ weak $\Rightarrow m$ only slightly modified
- reflection from a jet: $U_{jet} \geq \frac{N}{N} \left(1 - \frac{k}{\sqrt{k^2 + m^2}}\right) \Rightarrow m$ and $c_{gz}$ changes sign
- static instability: amplitude growing until static instability
- modulational instability: $|m| \approx |k| \Rightarrow$ wave packet is shrinking, its amplitude growing $\Rightarrow$ static instability
- critical layer: $U_{jet} \approx -c_p \Rightarrow m$ grows to infinity, wavepacket collapses
Numerical experiments

Induced wind, refraction by a jet ($\lambda_x = 10\text{km}$, $\lambda_z = 1\text{km}$)

*Rieper et al., 2013*

- **a):** LES
- **b):** conventional WKB raytracer
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SPARC Symposium, State College, 17 May 2016
Coupled gravity-wave raytracing
Numerical experiments

Wave energy, reflection from a jet ($\lambda_x = 10\, km$, $\lambda_z = 1\, km$)
Wave energy, partial reflection from a jet ($\lambda_x = 6 \text{km}, \lambda_z = 3 \text{km}$)
Numerical experiments

Static instability ($\lambda_x = \lambda_z = 1 km$)

- LES
- ELES
- Uin
- WKB-la
- E WKB-la
- WKB-la alpha=1.6
- green: wave energy
- blue: mean flow energy
- red: sum

Time (N x t)

Integrated energy

Altitude (km)
Conclusions

- **Caustics problem resolved**: comparison with LES shows that the WKB phase-space concept is working well in an *isothermal atmosphere like background* too.
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- **The Lagrangian raytracer is very efficient**: factor of 10-100 compared to the Eulerian model and factor of 1000-10000 compared to LES ⇒ the Lagrangian model is the main candidate for future work.
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- **The Lagrangian WKB model has been supplemented with a wave breaking parametrization**, which can slightly improve the fit to LES.
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- **However! the WKB model without the breaking parametrization works surprisingly well** → the most important part of the wave - background interaction is the weakly-nonlinear coupling and *NOT* wave breaking or wave-wave interactions.
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- However! the WKB model without the breaking parametrization works surprisingly well ⇒ the most important part of the wave - background interaction is the weakly-nonlinear coupling and *NOT* wave breaking or wave-wave interactions.

- **There is a good reason to try out fully coupled transient approaches** in GW parametrizations and go beyond the non-acceleration theorem applied in most current schemes.


Questions?
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<td>Refraction by a jet</td>
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<td>$n_z = 400, \Delta n = 70$</td>
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<td>$dz \approx 100\text{m}, \Delta m = 10^{-4} s^{-1}$</td>
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<td>$\lambda_a = 10\text{km}, \lambda_c = 1\text{km}$</td>
<td>$T = 300\text{K}$</td>
<td>$L_z = 40\text{km}$</td>
<td>$n_z = 400, \Delta n = 180$</td>
</tr>
<tr>
<td></td>
<td>$k = 2\pi/\lambda_a, m = 2\pi/\lambda_c$</td>
<td>$N \approx 0.018$</td>
<td>$m \in [-0.01, 0.008]$</td>
<td>$dz \approx 100\text{m}, \Delta m = 10^{-4} s^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$z_0 = 10\text{km}, \Delta z = 10\text{km}$</td>
<td>$u_0 = 40\text{m/s}$</td>
<td>WKB Lagrange:</td>
<td>WKB Lagrange:</td>
</tr>
<tr>
<td></td>
<td>branch $= -1, a_0 = 0.1$</td>
<td>$z_a = 25\text{km}$</td>
<td>$L_z = 40\text{km}$</td>
<td>$n_z = 400, \Delta z = 600\text{km}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\Delta a = 10\text{km}$</td>
<td>LES:</td>
<td>LES:</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$L_z = 40\text{km}, L_x = 10\text{km}$</td>
<td>$n_z = 2500, n_x = 64$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$dz = 16\text{m}, \Delta x = 156\text{m}$</td>
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<tr>
<td>REFRL</td>
<td>Cosine shape</td>
<td>non-Boussinesq</td>
<td>WKB Lagrange:</td>
<td>WKB Lagrange:</td>
</tr>
<tr>
<td>Partial Reflection from a jet</td>
<td>$\lambda_a = 6\text{km}, \lambda_c = 2\text{km}$</td>
<td>$T = 300\text{K}$</td>
<td>$L_z = 50\text{km}$</td>
<td>$n_z = 166, \Delta z = 1800\text{m}$</td>
</tr>
<tr>
<td></td>
<td>$k = 2\pi/\lambda_a, m = 2\pi/\lambda_c$</td>
<td>$N \approx 0.018$</td>
<td>$m \in [0.001, 0.008]$</td>
<td>$dz = 300\text{m}, \Delta m = 4320$</td>
</tr>
<tr>
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<td>$z_0 = 10\text{km}, \Delta z = 10\text{km}$</td>
<td>$u_0 = 9.75\text{m/s}$</td>
<td>LES:</td>
<td>LES:</td>
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<tr>
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<td>branch $= -1, a_0 = 0.1$</td>
<td>$z_a = 25\text{km}$</td>
<td>$L_z = 50\text{km}, L_x = 6\text{km}$</td>
<td>$n_z = 538, n_x = 32$</td>
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<td></td>
<td>$\Delta a = 10\text{km}$</td>
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<td>$dz = 93\text{m}, \Delta x = 187\text{m}$</td>
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<tr>
<td>Model</td>
<td>Gaussian shape</td>
<td>non-Boussinesq</td>
<td>WKB Lagrange</td>
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<tr>
<td>-------</td>
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</tr>
<tr>
<td>STIH Static</td>
<td>$\lambda_x = 30 \text{ km}$, $\lambda_z = 3 \text{ km}$</td>
<td>$T = 300 \text{ K}$</td>
<td>$L_z = 80 \text{ km}$</td>
<td></td>
</tr>
<tr>
<td>Instability</td>
<td>$k = 2\pi / \lambda_x$, $m = 2\pi / \lambda_z$</td>
<td>$N \approx 0.018$</td>
<td>$L_z = 80 \text{ km}, L_x = 30 \text{ km}$</td>
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<tr>
<td>Hydrostatic</td>
<td>$z_0 = 10 \text{ km}$, $\Delta z_p = 25 \text{ km}$</td>
<td>LES</td>
<td>$d_z \approx 94 \text{ m}, dx \approx 940 \text{ m}$</td>
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<tr>
<td>Wavepacket</td>
<td>$branch = -1, a_0 = 0.5$</td>
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<table>
<thead>
<tr>
<th>Model</th>
<th>Gaussian shape</th>
<th>non-Boussinesq</th>
<th>WKB Lagrange</th>
</tr>
</thead>
<tbody>
<tr>
<td>STINH Static</td>
<td>$\lambda_x = 1 \text{ km}$, $\lambda_z = 1 \text{ km}$</td>
<td>$T = 300 \text{ K}$</td>
<td>$L_z = 30 \text{ km}$</td>
</tr>
<tr>
<td>Instability</td>
<td>$k = 2\pi / \lambda_x$, $m = 2\pi / \lambda_z$</td>
<td>$N \approx 0.018$</td>
<td>$L_z = 30 \text{ km}, L_x = 1 \text{ km}$</td>
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<tr>
<td>Non-hydrostatic</td>
<td>$z_0 = 10 \text{ km}$, $\Delta z_p = 10 \text{ km}$</td>
<td>LES</td>
<td>$d_z \approx 31 \text{ m}, dx \approx 310 \text{ m}$</td>
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<tr>
<td>Wavepacket</td>
<td>$branch = -1, a_0 = 0.9$</td>
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<table>
<thead>
<tr>
<th>Model</th>
<th>Cosine shape</th>
<th>non-Boussinesq</th>
<th>WKB Lagrange</th>
</tr>
</thead>
<tbody>
<tr>
<td>MI Modulational</td>
<td>$\lambda_x = 1 \text{ km}$, $\lambda_z = 1 \text{ km}$</td>
<td>$T = 300 \text{ K}$</td>
<td>$L_z = 60 \text{ km}$</td>
</tr>
<tr>
<td>Instability</td>
<td>$k = 2\pi / \lambda_x$, $m = 2\pi / \lambda_z$</td>
<td>$N \approx 0.018$</td>
<td>$L_z = 60 \text{ km}, L_x = 1 \text{ km}$</td>
</tr>
<tr>
<td></td>
<td>$z_0 = 10 \text{ km}$, $\Delta z_p = 20 \text{ km}$</td>
<td>LES</td>
<td>$d_z \approx 31 \text{ m}, dx \approx 310 \text{ m}$</td>
</tr>
<tr>
<td></td>
<td>$branch = -1, a_0 = 0.1$</td>
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<table>
<thead>
<tr>
<th>Model</th>
<th>Cosine shape</th>
<th>non-Boussinesq</th>
<th>WKB Lagrange</th>
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</thead>
<tbody>
<tr>
<td>CL Critical Layer</td>
<td>$\lambda_x = 10 \text{ km}$, $\lambda_z = 1 \text{ km}$</td>
<td>$T = 300 \text{ K}$</td>
<td>$L_z = 30 \text{ km}$</td>
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<tr>
<td></td>
<td>$k = 2\pi / \lambda_x$, $m = 2\pi / \lambda_z$</td>
<td>$N \approx 0.018$</td>
<td>$L_z = 30 \text{ km}, L_x = 10 \text{ km}$</td>
</tr>
<tr>
<td></td>
<td>$z_0 = 10 \text{ km}$, $\Delta z_p = 10 \text{ km}$</td>
<td>$u_0 = -11 \text{ m/s}$</td>
<td>$d_z \approx 31 \text{ m}, dx \approx 310 \text{ m}$</td>
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<tr>
<td></td>
<td>$branch = -1, a_0 = 0.1$</td>
<td>$z_a = 25 \text{ km}$</td>
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<tr>
<td></td>
<td></td>
<td>$\Delta z_a = 10 \text{ km}$</td>
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