Stability of large-amplitude gravity waves with non-uniform stratification

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2D Euler equations with a multiple scale ansatz \((\varepsilon z = Z)\) separating the wave field

\[
U = (u, w, B, P)(x, z, t)
\]  \(1\)

from hydrostatic background \(N(Z)\) and \(\rho(Z)\)

\[
\begin{align*}
0 &= D_t u + \partial_x P + \varepsilon NB \partial_x P \\
0 &= D_t w + \partial_z P - NB - \varepsilon (N^2 P - NB \partial_z P) + \text{h.o.t.} \\
0 &= D_t B + Nw + \varepsilon (N^2 + d_Z \ln(N)) wB \\
0 &= \nabla \cdot \begin{pmatrix} u \\ w \end{pmatrix} + \varepsilon (N^2 + d_Z \ln(\rho)) w + \text{h.o.t.}
\end{align*}
\]  \(2\)
Governing equations

- 2D Euler equations with a multiple scale ansatz \( \epsilon z = Z \) separating the wave field

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0 = \nabla \cdot \begin{pmatrix} u \\ w \end{pmatrix} + \epsilon (N^2 + dZ \ln(\rho))w + \text{h.o.t.}
\]  

- with the distinguished limit \( \epsilon \approx \text{Ma} \approx \text{Fr}^2 \)
2D Euler equations with a multiple scale ansatz ($\varepsilon z = Z$) separating the wave field

$$U = (u, w, B, P)(x, z, t)$$

from hydrostatic background $N(Z)$ and $\rho(Z)$

$$0 = D_t u + \partial_x P + \varepsilon N B \partial_x P$$

$$0 = D_t w + \partial_z P - N B - \varepsilon (N^2 P - N B \partial_z P) + \text{h.o.t.}$$

$$0 = D_t B + N w + \varepsilon (N^2 + d_z \ln(N)) w B$$

$$0 = \nabla \cdot \left( \begin{array}{c} u \\ w \end{array} \right) + \varepsilon (N^2 + d_z \ln(\rho)) w + \text{h.o.t.}$$

with the distinguished limit $\varepsilon \approx \text{Ma} \approx \text{Fr}^2$

● Boussinesq Equations
Governing equations

- 2D Euler equations with a multiple scale ansatz ($\varepsilon z = Z$) separating the wave field

\[ U = (u, w, B, P)(x, z, t) \]  \hspace{1cm} (1)

from hydrostatic background $N(Z)$ and $\rho(Z)$

\[ 0 = D_t u + \partial_x P + \varepsilon NB \partial_x P \]
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\[ 0 = D_t B + Nw + \varepsilon (N^2 + d_z \ln(N)) wB \]
\[ 0 = \nabla \cdot \left( \begin{pmatrix} u \\ w \end{pmatrix} \right) + \varepsilon (N^2 + d_z \ln(\rho)) w + \text{h. o. t.} \]  \hspace{1cm} (2)

- with the distinguished limit $\varepsilon \approx \text{Ma} \approx \text{Fr}^2$

- *Boussinesq Equations*

- *Next Order Soundproof Equations* (Achatz et al., 2010)
Asymptotic scheme

- Large-amplitude, weakly nonlinear, spectral ansatz

\[ U = \sum_{n,m} \epsilon^n \hat{U}_{n,m} \exp\left(\frac{im\Phi}{\epsilon}\right) + o(\epsilon^M) \]  (3)

With additional WKB assumption \((\epsilon_x, \epsilon_z, \epsilon_t) \mapsto (X, Z, T)\) that amplitudes and phase depend only on the slow coordinates.

Injecting the ansatz into the Next Order Soundproof Equations results in equations of motion (EOM) for the ansatz functions. The set of EOM's is denoted as Modulational Equations inspired by AM/FM-radio.
Asymptotic scheme

- Large-amplitude, weakly nonlinear, spectral ansatz

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U = \sum_{n,m} \varepsilon^n \hat{U}_{n,m} \exp \left( im \frac{\Phi}{\varepsilon} \right) + o(\varepsilon^M)
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(3)

- With additional WKB assumption

\[(\varepsilon x, \varepsilon z, \varepsilon t) = (X, Z, T) \mapsto \hat{U}_{n,m}, \Phi\]

(4)

that amplitudes and phase depend only on the slow coordinates
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that amplitudes and phase depend only on the slow coordinates

- Injecting the ansatz into the *Next Order Soundproof Equations* results in equations of motion (EOM) for the ansatz functions

- The set of EOM’s is denoted as *Modulational Equations* inspired by AM/FM-radio
The modulational equations

- Dispersion relation

\[
(\partial_\tau \Phi + \partial_X \Phi \hat{u}_{0,0})^2 = \hat{\omega}^2(\nabla_X \Phi)
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(5)
The modulational equations

- Dispersion relation

\[(\partial_T \Phi + \partial_X \Phi \hat{u}_{0,0})^2 = \hat{\omega}^2 (\nabla_X \Phi)\]  

- Wave action density conservation

\[\partial_T \rho |A|^2 \hat{\omega} + \nabla_X \cdot \left( c_g \rho |A|^2 \right) = 0\]
The modulational equations

- **Dispersion relation**

\[(\partial_T \Phi + \partial_X \Phi \hat{u}_{0,0})^2 = \hat{\omega}^2 (\nabla_X \Phi)\]  \hspace{1cm} (5)

- **Wave action density conservation**

\[\partial_T \frac{\rho |A|^2}{\hat{\omega}} + \nabla_X \cdot \left( c_g \frac{\rho |A|^2}{\hat{\omega}} \right) = 0\]  \hspace{1cm} (6)

- **Induced mean flow**

\[\rho \partial_T \hat{u}_{0,0} + \rho \partial_X \hat{P}_{0,0} = -2 \nabla_X \cdot \left( k_x \frac{\partial \hat{\omega}}{\partial k} \frac{\rho |A|^2}{\hat{\omega}} \right)\]  \hspace{1cm} (7)

\[\partial_X \hat{u}_{0,0} = 0\]  \hspace{1cm} (8)
Is it possible to derive solutions $\bar{U}$ that are stationary in a translational coordinate system (wave trains)?

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} - CT$$  \hspace{1cm} (9)
Wave train solutions

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(9)

Yes! But only for 2 disjoint cases
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1. $C_z = 0$ (*Horizontally Propagating Wave Train*)
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- Yes! But only for 2 disjoint cases

1. \( C_z = 0 \) (*Horizontally Propagating Wave Train*)
2. Background is isothermal and \( C \) arbitrary (*Isothermal Wave Train*)
Wave train solutions

▶ Is it possible to derive solutions \( \overline{U} \) that are stationary in a translational coordinate system (wave trains)?

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▶ Yes! But only for 2 disjoint cases

1. \( C_z = 0 \) (Horizontally Propagating Wave Train)
2. Background is isothermal and \( C \) arbitrary (Isothermal Wave Train)

▶ For case 1 we were able to derive analytical solutions.
The horizontally propagating wave train (case 1)
Assuming horizontal homogeneity, the Modulational Equations in the isothermal case reduce to an ODE

\[
\frac{d\zeta}{d\zeta} \begin{pmatrix} a \\ k_z \end{pmatrix} = \vec{F}(a, k_z) \quad \text{where} \quad a = \frac{|A|^2}{\hat{\omega}}.
\]
Assuming horizontal homogeneity, the \textit{Modulational Equations} in the isothermal case reduce to an ODE

\[
d\zeta \left( \frac{a}{k_z} \right) = \vec{F}(a, k_z) \quad \text{where} \quad a = \frac{|A|^2}{\hat{\omega}}.
\]  

(10)

Phase portrait of the ODE  

Numerical results (explicit Euler scheme)

- Also locally confined wave packet
Transformation to curvilinear translational coordinates to obtain plain wave

\[ \Phi = \epsilon \varphi = K_x \xi + \int k_z(\zeta) \, d(\zeta) \]

\[ \Psi = \epsilon \psi = K_x \xi - \int \frac{K_x^2}{k_z(\zeta)} \, d(\zeta) \]
Stability of the wave trains

- Transformation to curvelinear translational coordinates to obtain plain wave

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- Linearize The Next Order Soundproof Equations in curvelinear translational coordinates at the wave train

\[ U(\varphi, \psi, t) = \bar{U}(\Phi, \Psi, e^{i\varphi}) + \tilde{U}(\varphi, \psi, t) \quad (11) \]
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Linearize The Next Order Soundproof Equations in curvelinear translational coordinates at the wave train

\[ U(\varphi, \psi, t) = \overline{U}(\Phi, \Psi, e^{i\varphi}) + \tilde{U}(\varphi, \psi, t) \quad (11) \]

Results in a multiple scale problem for the perturbation
WKB ansatz for the perturbation

\[ \tilde{U} = \hat{U}(\phi, \Psi) \exp \left( \frac{\Theta(\Psi)}{\varepsilon} - \sigma t \right) \]  

(12)
WKB ansatz for the perturbation

\[ \tilde{U} = \hat{U}(\varphi, \Psi) \exp \left( \frac{\Theta(\Psi)}{\varepsilon} - \sigma t \right) \]  \hspace{1cm} (12)

In the limit \( \varepsilon \to 0 \) and \( (\varphi, \Psi) \) fixed, this yields a *Floquet* system \( \mathcal{A} \hat{U} = \partial_{\varphi} \hat{U} \) with the matrix \( \mathcal{A}(\varphi) = \mathcal{A}(\varphi + 2\pi) \) being periodic.
WKB ansatz for the perturbation

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In the limit \( \varepsilon \to 0 \) and \((\varphi, \Psi)\) fixed, this yields a Floquet system \( \mathcal{A} \hat{U} = \partial_\varphi \hat{U} \) with the matrix \( \mathcal{A}(\varphi) = \mathcal{A}(\varphi + 2\pi) \) being periodic

Floquet theory provides a solution of the form

\[ \hat{U} = \sum_{m=-\infty}^{\infty} \tilde{U}_m(\Psi) e^{i(\gamma + m)\varphi} \text{ with } \gamma \text{ the Floquet parameter} \]  

(13)
Stability of the wave trains

- WKB ansatz for the perturbation

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\tilde{U} = \hat{U}(\varphi, \Psi) \exp \left( \frac{\Theta(\Psi)}{\varepsilon} - \sigma t \right)
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- In the limit \( \varepsilon \to 0 \) and \((\varphi, \Psi)\) fixed, this yields a Floquet system \( \mathcal{A}\hat{U} = \partial_{\varphi}\hat{U} \) with the matrix \( \mathcal{A}(\varphi) = \mathcal{A}(\varphi + 2\pi) \) being periodic

- Floquet theory provides a solution of the form

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\hat{U} = \sum_{m=-\infty}^{\infty} \tilde{U}_m(\Psi)e^{i(\gamma + m)\varphi} \text{ with } \gamma \text{ the Floquet parameter}
\]  

- Route to parametric instability (PSI). But our parameters are functions!
Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.
Conclusion

- Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.
- We found two classes of locally confined wave train solutions.
Conclusion

▶ Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.
▶ We found two classes of locally confined wave train solutions.
▶ For one class complete set of analytical solutions which may be used to test pseudo-incompressible numerical solver.
Conclusion

- Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.
- We found two classes of locally confined wave train solutions.
- For one class complete set of analytical solutions which may be used to test pseudo-incompressible numerical solver.
- Parametric instability theory for wave trains including induced mean-flow interaction.