Stability of large-amplitude gravity waves with non-uniform stratification

Mark Schlutow and Rupert Klein
Freie Universität Berlin

2016 SPARC Gravity Wave Symposium

- 2D Euler equations with a multiple scale ansatz ($\varepsilon z = Z$) separating the wave field

$$U = (u, w, B, P)(x, z, t) \tag{1}$$

from hydrostatic background $N(Z)$ and $\rho(Z)$

$$\begin{aligned}
0 &= D_t u + \partial_x P + \varepsilon N B \partial_x P \\
0 &= D_t w + \partial_z P - NB - \varepsilon (N^2 P - NB\partial_z P) + \text{h.o.t.} \\
0 &= D_t B + Nw + \varepsilon (N^2 + d_Z \ln(N)) wB \\
0 &= \nabla \cdot \begin{pmatrix} u \\ w \end{pmatrix} + \varepsilon (N^2 + d_Z \ln(\rho)) w + \text{h.o.t.}
\end{aligned} \tag{2}$$

- 2D Euler equations with a multiple scale ansatz ($\varepsilon z = Z$) separating the wave field

$$U = (u, w, B, P)(x, z, t) \tag{1}$$

from hydrostatic background $N(Z)$ and $\rho(Z)$

$$\begin{aligned}
0 =& D_t u + \partial_x P + \varepsilon N B \partial_x P \\
0 =& D_t w + \partial_z P - NB - \varepsilon(N^2 P - NB\partial_z P) + \mathrm{h.o.t.} \\
0 =& D_t B + Nw + \varepsilon(N^2 + d_Z \ln(N)) wB \\
0 =& \nabla \cdot \begin{pmatrix} u \\ w \end{pmatrix} + \varepsilon(N^2 + d_Z \ln(\rho)) w + \mathrm{h.o.t.}
\end{aligned} \tag{2}$$

- with the distinguished limit $\varepsilon \approx \mathrm{Ma} \approx \mathrm{Fr}^2$

- 2D Euler equations with a multiple scale ansatz ($\varepsilon z = Z$) separating the wave field

$$U = (u, w, B, P)(x, z, t) \tag{1}$$

from hydrostatic background $N(Z)$ and $\rho(Z)$

$$\begin{aligned}
0 =& D_t u + \partial_x P + \varepsilon NB\partial_x P \\
0 =& D_t w + \partial_z P - NB - \varepsilon(N^2 P - NB\partial_z P) + \text{h. o. t.} \\
0 =& D_t B + Nw + \varepsilon(N^2 + d_Z \ln(N))wB \\
0 =& \nabla \cdot \begin{pmatrix} u \\ w \end{pmatrix} + \varepsilon(N^2 + d_Z \ln(\rho))w + \text{h. o. t.}
\end{aligned} \tag{2}$$

- with the distinguished limit $\varepsilon \approx \mathrm{Ma} \approx \mathrm{Fr}^2$
- *Boussinesq Equations*

- 2D Euler equations with a multiple scale ansatz ($\varepsilon z = Z$) separating the wave field

$$U = (u, w, B, P)(x, z, t) \tag{1}$$

from hydrostatic background $N(Z)$ and $\rho(Z)$

$$\begin{aligned}
0 =& D_t u + \partial_x P + \varepsilon N B \partial_x P \\
0 =& D_t w + \partial_z P - NB - \varepsilon(N^2 P - NB\partial_z P) + \text{h.o.t.} \\
0 =& D_t B + Nw + \varepsilon(N^2 + d_Z \ln(N)) wB \\
0 =& \nabla \cdot \begin{pmatrix} u \\ w \end{pmatrix} + \varepsilon(N^2 + d_Z \ln(\rho)) w + \text{h.o.t.}
\end{aligned} \tag{2}$$

- with the distinguished limit $\varepsilon \approx \mathrm{Ma} \approx \mathrm{Fr}^2$
- *Boussinesq Equations*
- *Next Order Soundproof Equations* (Achatz et al., 2010)

- Large-amplitude, weakly nonlinear, spectral ansatz

$$U = \sum_{n,m} \varepsilon^n \hat{U}_{n,m} \exp\left(im \frac{\Phi}{\varepsilon} \right) + o(\varepsilon^M) \tag{3}$$

- Large-amplitude, weakly nonlinear, spectral ansatz

$$U = \sum_{n,m} \varepsilon^n \hat{U}_{n,m} \exp\left(im\frac{\Phi}{\varepsilon}\right) + o(\varepsilon^M) \tag{3}$$

- With additional WKB assumption

$$(\varepsilon x, \varepsilon z, \varepsilon t) = (X, Z, T) \mapsto \hat{U}_{n,m}, \Phi \tag{4}$$

that amplitudes and phase depend only on the slow coordinates

- Large-amplitude, weakly nonlinear, spectral ansatz

$$U = \sum_{n,m} \varepsilon^n \hat{U}_{n,m} \exp\left(im\frac{\Phi}{\varepsilon}\right) + o(\varepsilon^M) \tag{3}$$

- With additional WKB assumption

$$(\varepsilon x, \varepsilon z, \varepsilon t) = (X, Z, T) \mapsto \hat{U}_{n,m}, \Phi \tag{4}$$

  that amplitudes and phase depend only on the slow coordinates
- Injecting the ansatz into the *Next Order Soundproof Equations* results in equations of motion (EOM) for the ansatz functions

- Large-amplitude, weakly nonlinear, spectral ansatz

$$U = \sum_{n,m} \varepsilon^n \hat{U}_{n,m} \exp\left(im \frac{\Phi}{\varepsilon} \right) + o(\varepsilon^M) \tag{3}$$

- With additional WKB assumption

$$(\varepsilon x, \varepsilon z, \varepsilon t) = (X, Z, T) \mapsto \hat{U}_{n,m}, \Phi \tag{4}$$

  that amplitudes and phase depend only on the slow coordinates
- Injecting the ansatz into the *Next Order Soundproof Equations* results in equations of motion (EOM) for the ansatz functions
- The set of EOM's is denoted as *Modulational Equations* inspired by AM/FM-radio

- Dispersion relation

$$(\partial_T \Phi + \partial_X \Phi \hat{u}_{0,0})^2 = \hat{\omega}^2 (\nabla_X \Phi) \tag{5}$$

- Dispersion relation

$$\left(\partial_T \Phi + \partial_X \Phi \hat{u}_{0,0}\right)^2 = \hat{\omega}^2 \left(\nabla_X \Phi\right) \tag{5}$$

- Wave action density conservation

$$\partial_T \frac{\rho |A|^2}{\hat{\omega}} + \nabla_X \cdot \left(\mathbf{c}_g \frac{\rho |A|^2}{\hat{\omega}} \right) = 0 \tag{6}$$

- Dispersion relation

$$\left(\partial_T \Phi + \partial_X \Phi \hat{u}_{0,0}\right)^2 = \hat{\omega}^2 (\nabla_X \Phi) \tag{5}$$

- Wave action density conservation

$$\partial_T \frac{\rho |A|^2}{\hat{\omega}} + \nabla_X \cdot \left(\mathbf{c}_g \frac{\rho |A|^2}{\hat{\omega}} \right) = 0 \tag{6}$$

- Induced mean flow

$$\rho \partial_T \hat{u}_{0,0} + \rho \partial_X \hat{P}_{0,0} = -2 \nabla_X \cdot \left(k_x \frac{\partial \hat{\omega}}{\partial \mathbf{k}} \frac{\rho |A|^2}{\hat{\omega}} \right) \tag{7}$$

$$\partial_X \hat{u}_{0,0} = 0 \tag{8}$$

- *Is it possible to derive solutions $\overline{U}$ that are stationary in a translational coordinate system (wave trains)?*

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} - \mathbf{c}T \tag{9}$$

- *Is it possible to derive solutions $\overline{U}$ that are stationary in a translational coordinate system (wave trains)?*

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} - \mathbf{c}T \tag{9}$$

- *Yes!* But only for 2 disjoint cases

- *Is it possible to derive solutions $\overline{U}$ that are stationary in a translational coordinate system (wave trains)?*

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} - \mathbf{C}T \tag{9}$$

- *Yes!* But only for 2 disjoint cases

1. $C_z = 0$ (*Horizontally Propagating Wave Train*)

- *Is it possible to derive solutions $\overline{U}$ that are stationary in a translational coordinate system (wave trains)?*

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} - \mathbf{C}T \tag{9}$$

- *Yes!* But only for 2 disjoint cases

1. $C_z = 0$ (*Horizontally Propagating Wave Train*)
2. Background is isothermal and $\mathbf{C}$ arbitrary (*Isothermal Wave Train*)

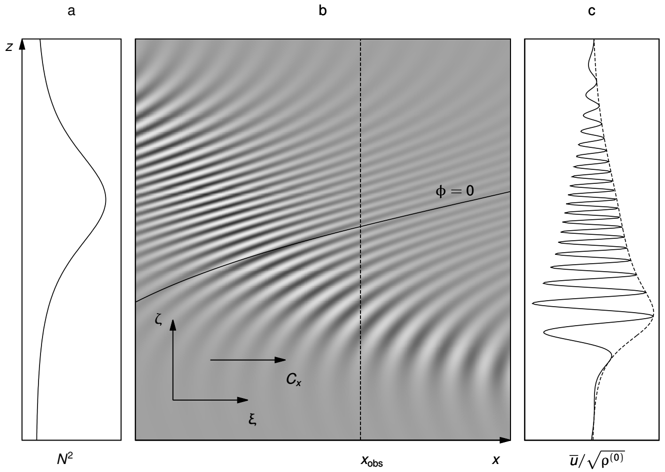
- *Is it possible to derive solutions* $\overline{U}$ *that are stationary in a translational coordinate system (wave trains)?*

$$\begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} - \mathbf{C}T \tag{9}$$

- *Yes!* But only for 2 disjoint cases

1. $C_z = 0$ (*Horizontally Propagating Wave Train*)
2. Background is isothermal and $\mathbf{C}$ arbitrary (*Isothermal Wave Train*)

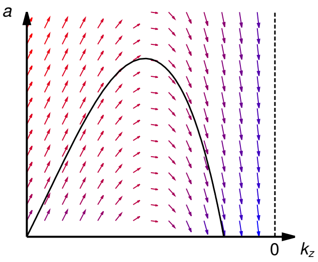
- For case 1 we were able to derive analytical solutions.


a
b
c
z
$\phi = 0$
ζ
C_x
ξ
N^2
x_{obs}
x
$\overline{u}/\sqrt{\rho^{(0)}}$

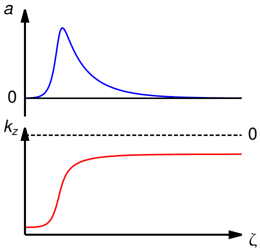
- Assuming horizontal homogeneity, the *Modulational Equations* in the isothermal case reduce to an ODE

$$d_\zeta \begin{pmatrix} a \\ k_z \end{pmatrix} = \vec{F}(a, k_z) \quad \text{where} \quad a = \frac{|A|^2}{\hat{\omega}}. \tag{10}$$

Phase portrait of the ODE



Numerical results (explicit Euler scheme)

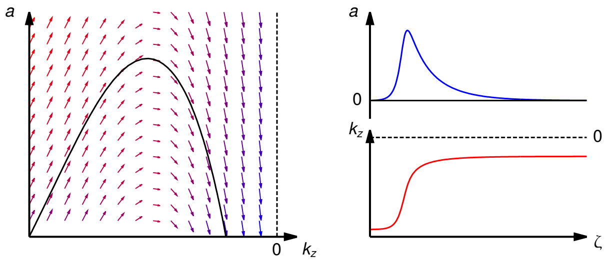


- Assuming horizontal homogeneity, the *Modulational Equations* in the isothermal case reduce to an ODE

$$d_\zeta \begin{pmatrix} a \\ k_z \end{pmatrix} = \vec{F}(a, k_z) \quad \text{where} \quad a = \frac{|A|^2}{\hat{\omega}}. \tag{10}$$

Phase portrait of the ODE

Numerical results (explicit Euler scheme)

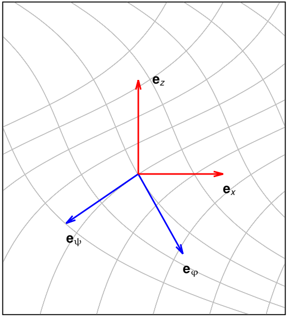


- Also locally confined wave packet

- Transformation to curvelinear translational coordinates to obtain plain wave

$$\Phi = \varepsilon\varphi = K_x \xi + \int \overline{k}_z(\zeta)\, d(\zeta)$$

$$\Psi = \varepsilon\psi = K_x \xi - \int \frac{K_x^2}{\overline{k}_z(\zeta)}\, d(\zeta)$$



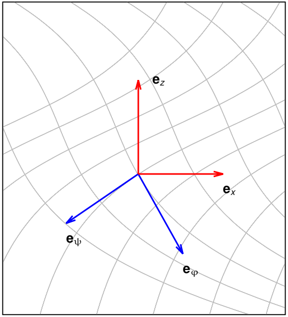
- Transformation to curvelinear translational coordinates to obtain plain wave

$$\Phi = \varepsilon\varphi = K_x\xi + \int \overline{k}_z(\zeta)\, d(\zeta)$$

$$\Psi = \varepsilon\psi = K_x\xi - \int \frac{K_x^2}{\overline{k}_z(\zeta)}\, d(\zeta)$$

- Linearize The Next Order Soundproof Equations in curvelinear translational coordinates at the wave train

$$U(\varphi, \psi, t) = \overline{U}(\Phi, \Psi, e^{i\varphi}) + \widetilde{U}(\varphi, \psi, t) \quad (11)$$



- Transformation to curvelinear translational coordinates to obtain plain wave

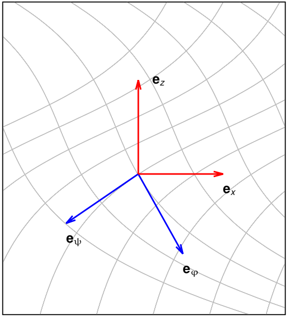
$$\Phi = \varepsilon\varphi = K_x\xi + \int \overline{k}_z(\zeta)\, d(\zeta)$$

$$\Psi = \varepsilon\psi = K_x\xi - \int \frac{K_x^2}{\overline{k}_z(\zeta)}\, d(\zeta)$$

- Linearize The Next Order Soundproof Equations in curvelinear translational coordinates at the wave train

$$U(\varphi,\psi,t) = \overline{U}(\Phi,\Psi,e^{i\varphi}) + \widetilde{U}(\varphi,\psi,t) \quad (11)$$

- Results in a multiple scale problem for the perturbation



- WKB ansatz for the perturbation

$$\widetilde{U} = \hat{\hat{U}}(\varphi, \Psi) \exp\left(\frac{\Theta(\Psi)}{\varepsilon} - \sigma t\right) \tag{12}$$

- WKB ansatz for the perturbation

$$\widetilde{U} = \hat{\hat{U}}(\varphi, \Psi) \exp\left(\frac{\Theta(\Psi)}{\varepsilon} - \sigma t\right) \tag{12}$$

- In the limit $\varepsilon \to 0$ and (φ, Ψ) fixed, this yields a *Floquet* system $\mathcal{A}\hat{\hat{U}} = \partial_\varphi \hat{\hat{U}}$ with the matrix $\mathcal{A}(\varphi) = \mathcal{A}(\varphi + 2\pi)$ being periodic

- WKB ansatz for the perturbation

$$\widetilde{U} = \hat{\widetilde{U}}(\varphi, \Psi) \exp\left(\frac{\Theta(\Psi)}{\varepsilon} - \sigma t\right) \tag{12}$$

- In the limit $\varepsilon \to 0$ and (φ, Ψ) fixed, this yields a *Floquet* system $\mathcal{A}\hat{\widetilde{U}} = \partial_\varphi \hat{\widetilde{U}}$ with the matrix $\mathcal{A}(\varphi) = \mathcal{A}(\varphi + 2\pi)$ being periodic
- *Floquet* theory provides a solution of the form

$$\hat{\widetilde{U}} = \sum_{m=-\infty}^{\infty} \widetilde{\widetilde{U}}_m(\Psi) e^{i(\gamma+m)\varphi} \text{ with } \gamma \text{ the Floquet parameter} \tag{13}$$

- WKB ansatz for the perturbation

$$\widetilde{U} = \hat{\hat{U}}(\varphi, \Psi) \exp\left(\frac{\Theta(\Psi)}{\varepsilon} - \sigma t\right) \tag{12}$$

- In the limit $\varepsilon \to 0$ and (φ, Ψ) fixed, this yields a *Floquet* system $\mathcal{A}\hat{\hat{U}} = \partial_\varphi \hat{\hat{U}}$ with the matrix $\mathcal{A}(\varphi) = \mathcal{A}(\varphi + 2\pi)$ being periodic
- *Floquet* theory provides a solution of the form

$$\hat{\hat{U}} = \sum_{m=-\infty}^{\infty} \widetilde{\widetilde{U}}_m(\Psi) e^{i(\gamma+m)\varphi} \text{ with } \gamma \text{ the Floquet parameter} \tag{13}$$

- Route to parametric instability (PSI). But our parameters are functions!

- Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.

- Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.
- We found two classes of locally confined wave train solutions.

- Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.
- We found two classes of locally confined wave train solutions.
- For one class complete set of analytical solutions which may be used to test pseudo-incompressible numerical solver.

- Weakly nonlinear theory for large-amplitude GW near breaking level facing non-uniform stratification.
- We found two classes of locally confined wave train solutions.
- For one class complete set of analytical solutions which may be used to test pseudo-incompressible numerical solver.
- Parametric instability theory for wave trains including induced mean-flow interaction.

Achatz, U., Klein, R.,Senf, F. (2010). Gravity waves, scale asymptotics and the pseudo-incompressible equations. *J. Fluid Mech., 663, 120-147*.

P. N. Lombard and J. J. Riley (1996). Instability and breakdown of internal gravity waves. I. Linear stability analysis *Phys. Fluids, Vol. 8, No. 12, December 1996*.