A wave-vortex decomposition method for one-dimensional spectra in the atmosphere and ocean

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A WORD FROM OUR FOUNDER

Letter from Jule Charney, 1947

February 12 1947

Lieutenant Philip D. Thompson
Institute for Advanced Study
Princeton, New Jersey

Dear Phil,

In the terminology which you graciously describe to me we might say that the atmosphere is a musical instrument on which one can play many tunes. High notes are sound waves, low notes are long inertial waves, and nature is a musician more of the Beethoven than of the Chopin type. He much prefers the low notes and only occasionally plays arpeggios in the treble and then only with a light hand. The oceans and the continents are the elephants in Saint-Saens' animal suite, marching in a slow cumbrous rhythm, one step every day or so. Of course, there are overtones; sound waves, billow clouds (gravity waves), inertial oscillations, etc., but these are unimportant and are heard only at N.Y.U. and M.I.T.

A tale of ~~two~~ three papers

JFM 2014. Lays out the theory and applies it to ocean data

PNAS 2014 & JAS 2016 submitted.
Gage-Nastrom spectrum of atmospheric data (Jörn Callies's talk)

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Wave–vortex decomposition of one-dimensional ship-track data

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Transition from geostrophic turbulence to inertia–gravity waves in the atmospheric energy spectrum

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The dynamics of mesoscale winds in the upper troposphere and lower stratosphere

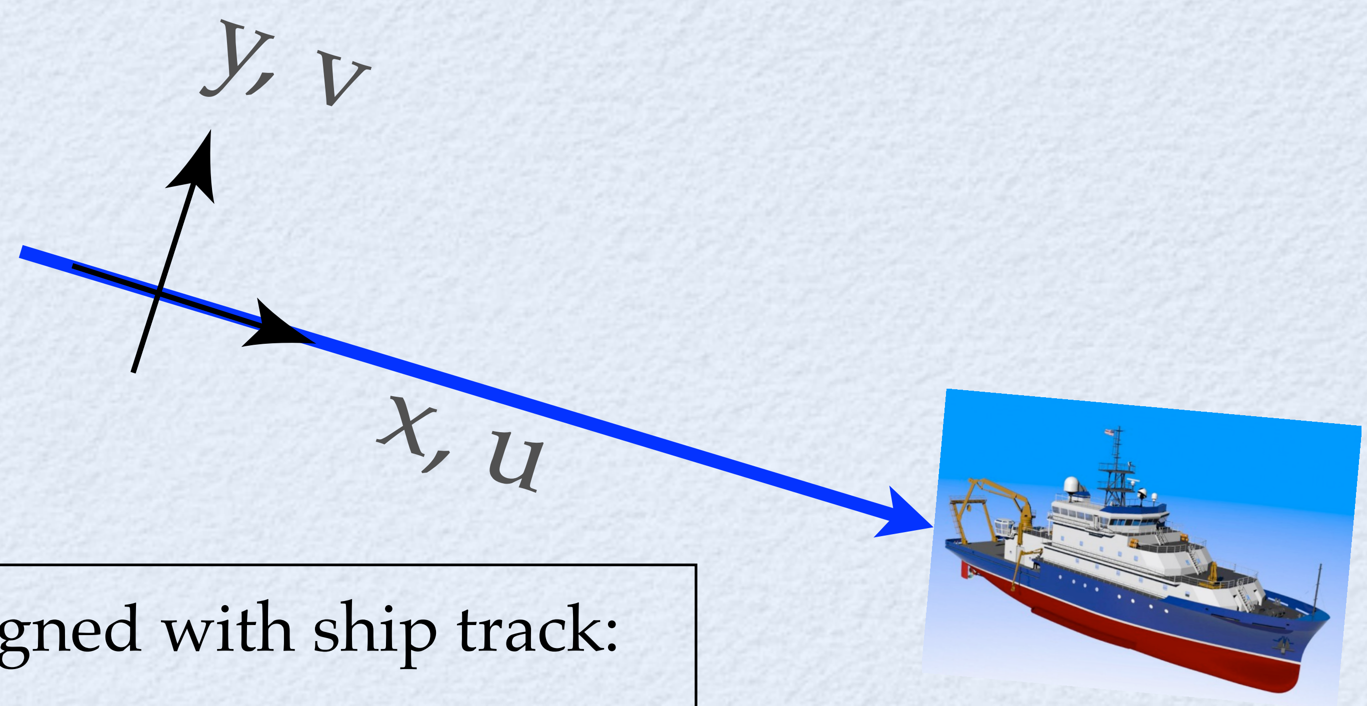
Jörn Callies[1,*], Oliver Bühler[2], and Raffaele Ferrari[1]

Flow analysis and one-dimensional data

Utility of one-dimensional data samples:

1. there is no other data
2. a local snapshot is sought.

Ship track measurements along x at fixed (y,z,t).
Towed submerged instruments



Ship track, flight track,rapid flow flow past fixed sensor…

Seeing reality only through a keyhole, or only via its shadow on the wall..

Aligned with ship track:

Longitudinal velocity u
Transversal velocity v

Kinematic and dynamic aliasing

Wave-Vortex Jigsaw Puzzle

Different dynamical processes can produce identical 1d spectra

In particular, internal waves and quasi-geostrophic vortices may be aliased together in the observations

Seeking the fingerprint of waves and vortices in snapshots of 1d data

Statistical assumptions:

a. horizontal isotropy
b. 3d homogeneity

Obviously strong.

One-dimensional Spectra

Random zero-mean homogeneous isotropic function defined in (xy)-plane observed along a ship track y=0: $\psi(x, y)$ $\mathbb{E}[\psi] = 0$

2d covariance

$$C^{\psi}(x, y) = \mathbb{E}[\psi(x_0, y_0)\psi(x_0 + x, y_0 + y)] = F(r) \qquad r = \sqrt{x^2 + y^2}$$

2d power spectrum

(Hankel transform pair)

$$\hat{C}^{\psi}(k, l) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C^{\psi}(x, y) e^{-i(kx+ly)}\, dx dy = \hat{F}(k_h) \qquad k_h = \sqrt{k^2 + l^2}.$$

1d ship track covariance and spectrum

$$\left. \begin{aligned} C^{\psi}(x) &= C^{\psi}(x, 0) = F(x) \\ \hat{C}^{\psi}(k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{C}^{\psi}(k, l)\, dl \end{aligned} \right\} \quad \boxed{\hat{C}^{\psi}(k) = \frac{1}{\pi} \int_{|k|}^{\infty} \frac{\hat{F}(k_h)}{\sqrt{k_h^2 - k^2}} k_h \mathrm{d}k_h.}$$

Shows kinematic aliasing

2D incompressible flow

Link between longitudinal and transverse spectra for incompressible flow.

Batchelor 1953, Charney 1971

$$u_x + v_y = 0 \Rightarrow u = -\psi_y \quad \text{and} \quad v = +\psi_x.$$

Velocity covariances

$$C^u(x,y) = -C^\psi_{yy}(x,y) \qquad C^v(x,y) = -C^\psi_{xx}(x,y)$$

Velocity spectra

$$\boxed{\hat{C}^u(k,l) = l^2\,\hat{C}^\psi(k,l) = l^2\,\hat{F}(k_h) \quad \text{and} \quad \hat{C}^v(k,l) = k^2\,\hat{C}^\psi(k,l) = k^2\,\hat{F}(k_h).}$$

1d velocity spectra

$$\begin{aligned}
\hat{C}^u(k) &= \frac{1}{\pi}\int_{|k|}^{\infty} \hat{F}(k_h)\sqrt{k_h^2 - k^2}\,k_h \mathrm{d}k_h,\\
\hat{C}^v(k) &= \frac{k^2}{\pi}\int_{|k|}^{\infty} \frac{\hat{F}(k_h)}{\sqrt{k_h^2 - k^2}}\,k_h \mathrm{d}k_h,
\end{aligned}$$

Horizontal velocity spectra are not isotropic, not equal, and not independent.

INCOMPRESSIBILITY FINGERPRINT

Link between longitudinal and transverse spectra for incompressible flow.
Batchelor 1953, Charney 1971

1d velocity spectra

$$\left.\begin{aligned}\psi \text{ only: } \hat{C}^u(k) &= \frac{1}{\pi}\int_{|k|}^{\infty} \hat{F}(k_h)\sqrt{k_h^2 - k^2}\, k_h \mathrm{d}k_h, \\ \hat{C}^v(k) &= \frac{k^2}{\pi}\int_{|k|}^{\infty} \frac{\hat{F}(k_h)}{\sqrt{k_h^2 - k^2}}\, k_h \mathrm{d}k_h,\end{aligned}\right\} \quad \boxed{\hat{C}^v(k) = -k\frac{d}{dk}\hat{C}^u(k)}$$

Provides simple test for layerwise non-divergent flow observed along a line!

Power law special case

$$\hat{C}^u(k), \hat{C}^v(k) \propto k^{-n}$$

$$\hat{C}^v(k) = n\hat{C}^u(k)$$

This was exploited in
Callies & Ferrari 13, JPO

Transversal velocity spectrum dominates if n>1, clearly so if n=3.

General case for uncorrelated stream function and potential

General Helmholtz decomposition in 2d (unique for periodic functions)

$$u = -\psi_y + \phi_x \quad \text{and} \quad v = \psi_x + \phi_y,$$

Key assumption renders velocity spectra additive

$$\mathbb{E}[\psi(x_0, y_0)\phi(x_0 + x, y_0 + y)] = 0$$

$$\boxed{\begin{aligned}\hat{C}^u(k) &= \frac{1}{\pi}\int_{|k|}^{\infty}\left[\hat{F}(k_h)\sqrt{k_h^2 - k^2} + \frac{k^2\hat{G}(k_h)}{\sqrt{k_h^2 - k^2}}\right] k_h \mathrm{d}k_h,\\ \hat{C}^v(k) &= \frac{1}{\pi}\int_{|k|}^{\infty}\left[\frac{k^2\hat{F}(k_h)}{\sqrt{k_h^2 - k^2}} + \hat{G}(k_h)\sqrt{k_h^2 - k^2}\right] k_h \mathrm{d}k_h.\end{aligned}}$$

No obvious differential relation between (u,v)-spectra

Two useful auxiliary spectral functions

$$\left.\begin{aligned}D^{\psi} &= \frac{1}{\pi}\int_{|k|}^{\infty}\hat{F}(k_h)\sqrt{k_h^2 - k^2}\, k_h\, dk_h\\ D^{\phi} &= \frac{1}{\pi}\int_{|k|}^{\infty}\hat{G}(k_h)\sqrt{k_h^2 - k^2}\, k_h\, dk_h\end{aligned}\right\}$$

$$\boxed{\begin{aligned}\hat{C}^u(k) &= D^{\psi}(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D^{\phi}(k)\\ \hat{C}^v(k) &= D^{\phi}(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D^{\psi}(k).\end{aligned}}$$

Helmholtz decomposition

The D-functions can easily be found from the observed velocity spectra!

$$\hat{C}^u(k) = D^\psi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D^\phi(k)$$

$$\hat{C}^v(k) = D^\phi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D^\psi(k).$$

Solve as coupled set of ODEs backwards in k using robust decay condition $D^\psi(+\infty) = D^\phi(+\infty) = 0$

Helmholtz decomposition of the kinetic energy spectrum (precursor Lindborg 07)

$$\frac{1}{2}\left[\hat{C}^u(k) + \hat{C}^v(k)\right] = \frac{1}{2}\left[D^\psi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D^\psi(k)\right] + \frac{1}{2}\left[D^\phi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D^\phi(k)\right]$$

rotational, stream function part

divergent, potential part

In practice: $S^u(k), S^v(k) \quad \Rightarrow \quad K(k) = K^\psi(k) + K^\phi(k)$

Gulf Stream Example

CF13, BCF14. Gulf Stream Data: New York Harbour to Bermuda at depth of z = 150 metres

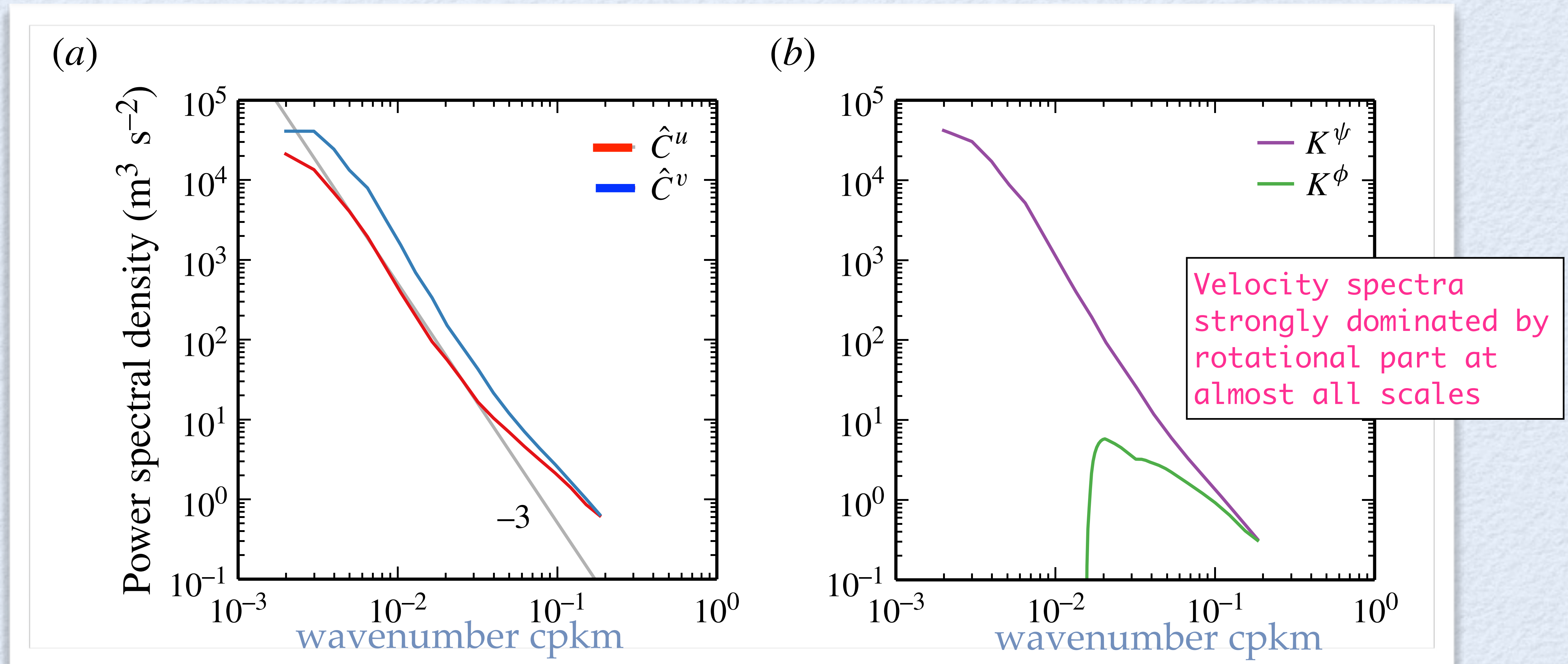


FIGURE 2. Observations from the Gulf Stream region: (*a*) observed transverse and longitudinal kinetic energy spectra $\hat{C}^u$ and $\hat{C}^v$; (*b*) decomposition into rotational and divergent components D^ψ and D^ϕ from (2.30), (2.31) and (2.27); here $K^\psi = (D^\psi - k\mathrm{d}D^\psi/\mathrm{d}k)/2$ and $K^\phi = (D^\phi - k\mathrm{d}D^\phi/\mathrm{d}k)/2$; (*c*) diagnosis of the balanced components of

Fingerprint of inertia-gravity waves

Linear Boussinesq equations in 3d with constant f and N

$$\boldsymbol{u}_t + f\widehat{\boldsymbol{z}} \times \boldsymbol{u} + \boldsymbol{\nabla} P = b\widehat{\boldsymbol{z}} \qquad b_t + N^2 w = 0 \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

$$(v_x - u_y)_t = f\, w_z = -f(u_x + v_y) \Rightarrow \psi_t + f\phi = 0.$$

For plane waves exp(i[kx + ly + mz - omega t]) this implies

$$\omega^2 = \frac{k_h^2 N^2 + m^2 f^2}{k_h^2 + m^2} \qquad \boxed{\hat{C}_W^{\psi} = \frac{f^2}{\omega^2}\hat{C}_W^{\phi}}$$

Can show that psi and phi are uncorrelated for isotropic stationary wave fields so the Helmholtz decomposition works. For propagating waves with omega > f the divergent part dominates the rotational part, but the ratio depends on the unobserved frequency.

However, rather more can be said....

Energy equipartition for IG waves

Assume random field composed of plane waves, which implies vertical homogeneity (this is a weak point for large-scale IG waves in the ocean)

$$b_t + N^2 w = 0$$

$$\phi_{xx} + \phi_{yy} = -w_z$$

Dispersion relation

$$\omega^2 = \frac{k_h^2 N^2 + m^2 f^2}{k_h^2 + m^2}$$

Easy to derive that (b/N)

$$\omega^2 \hat{C}_W^b = N^2 \hat{C}_W^w \quad \text{and} \quad m^2 \hat{C}_W^w = k_h^4 \hat{C}_W^\phi.$$

This implies an equipartition relation for 3d spectra (no dependence on m):

$$\hat{C}_W^b(k, l, m) + k_h^2 \hat{C}_W^\psi(k, l, m) = k_h^2 \hat{C}_W^\phi(k, l, m) + \hat{C}_W^w(k, l, m).$$

potential rotational KE divergent KE vertical KE

Corresponding relation for 1d IGW spectra:

$$\boxed{\hat{C}_W^b(k) + \left[D_W^\psi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D_W^\psi(k)\right] = \left[D_W^\phi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D_W^\phi(k)\right] + \hat{C}_W^w(k).}$$

Hydrostatic
Wave energy spectrum

Vertical velocity is typically not observed with present technology, but is negligible compared to horizontal velocity for hydrostatic waves (good assumption, not perfect):

$$w_t = b - P_z \approx 0 \qquad \omega^2 \ll N^2 \qquad k_h^2 \ll m^2$$

Can neglect vertical kinetic energy

$$\boxed{\hat{C}_W^w(k) \approx 0}$$

Hydrostatic equipartition relation now links potential and horizontal KE:

$$\boxed{\hat{C}_W^b(k) + \left[D_W^\psi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D_W^\psi(k)\right] = \left[D_W^\phi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D_W^\phi(k)\right]}$$

Hydrostatic wave energy spectrum in 1d

$$E_W(k) = \tfrac{1}{2}\left[\hat{C}_W^u(k) + \hat{C}_W^v(k) + \hat{C}_W^b(k)\right] \quad \Rightarrow \quad \boxed{E_W(k) = D_W^\phi(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D_W^\phi(k).}$$

Wave energy spectrum can be computed from horizontal velocity spectra only!!

Story so far

Observe

$$\hat{C}^u(k),\ \hat{C}^v(k)$$

Helmholtz
decomposition

Deduce/predict

$$D^\psi(k),\ D^\phi(k)$$

$$D^\phi_W(k) = D^\phi(k)$$

Wave energy predicted
from this!

$$\boxed{E_W(k) = D^\phi_W(k) - k\frac{\mathrm{d}}{\mathrm{d}k}D^\phi_W(k).}$$

Still unknown

$$D^\psi_W(k) + D^\psi_V(k) = D^\psi(k)$$

Unknown partition of wave-vortex mixture for rotational component (3>2)

Need to either know frequency of waves or observe the buoyancy field

Buoyancy observations

If buoyancy is observed then $\hat{C}^b(k)$ is known!

Good news:

Total energy spectrum observed, wave energy spectrum deduced

$$E(k) = E_W(k) + E_V(k) = \frac{1}{2}\left[\hat{C}^u(k) + \hat{C}^v(k) + \hat{C}^b(k)\right]$$

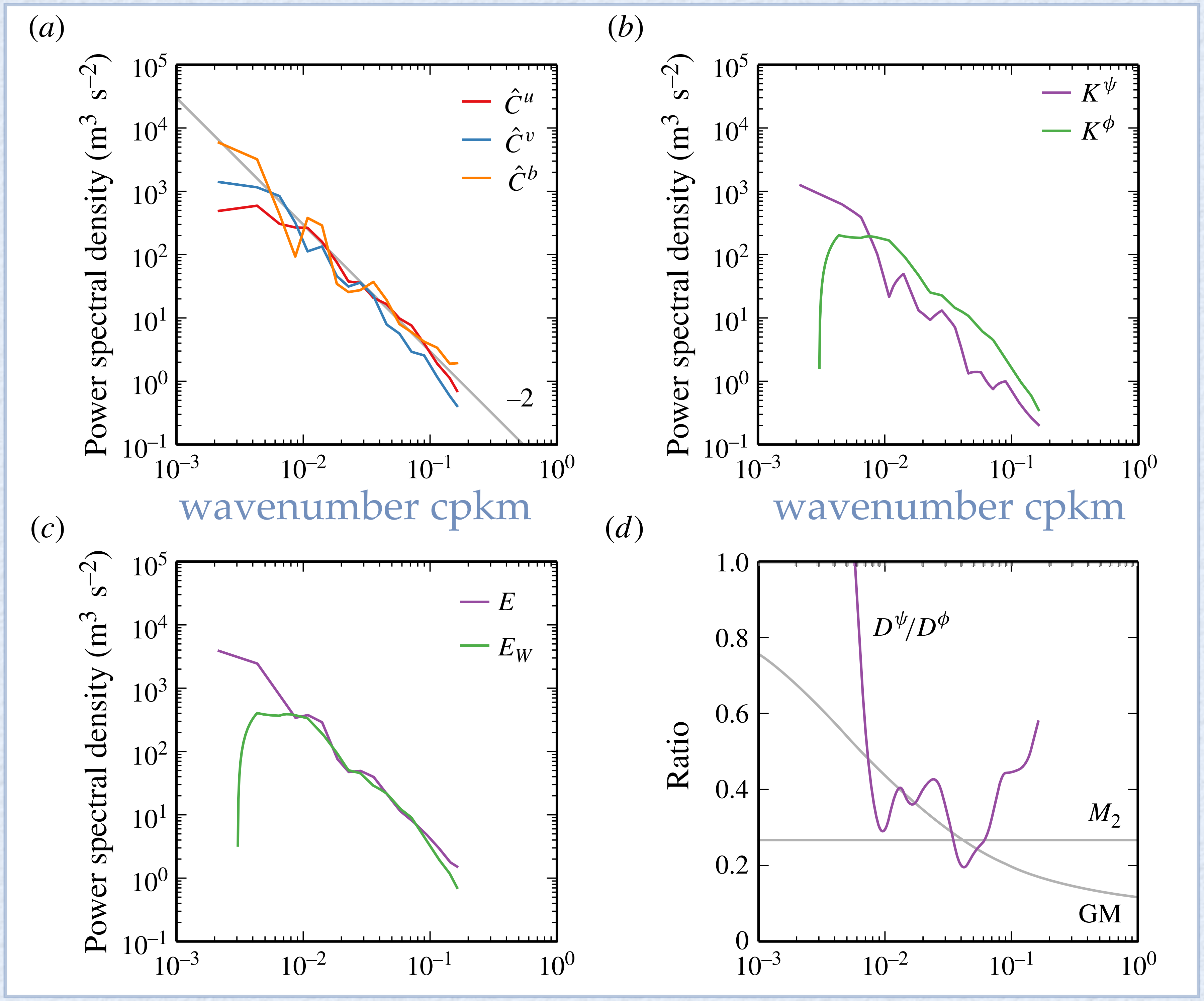
Residual vortex energy spectrum now known

$$E_V(k) = \frac{1}{2}\left[\hat{C}^u_V(k) + \hat{C}^v_V(k) + \hat{C}^b_V(k)\right] = E(k) - E_W(k)$$

A complete wave-vortex decomposition of energy has been achieved

Eastern Subtropical North Pacific

CF13, BCF14. Horizontal velocities and buoyancy at depth of z = 200 metres



Spectra flatter than Gulf Stream data

Divergent component important below 100 km

Wave spectrum takes over in energy budget at same scale

Compared D^psi/D^phi to GM model for waves and value of M2 tide

Waves clearly dominate at scales smaller than 100 km, with likely important tidal component

Gage-Nastrom Saga

Nastrom Gage 85, Gage Nastrom 86

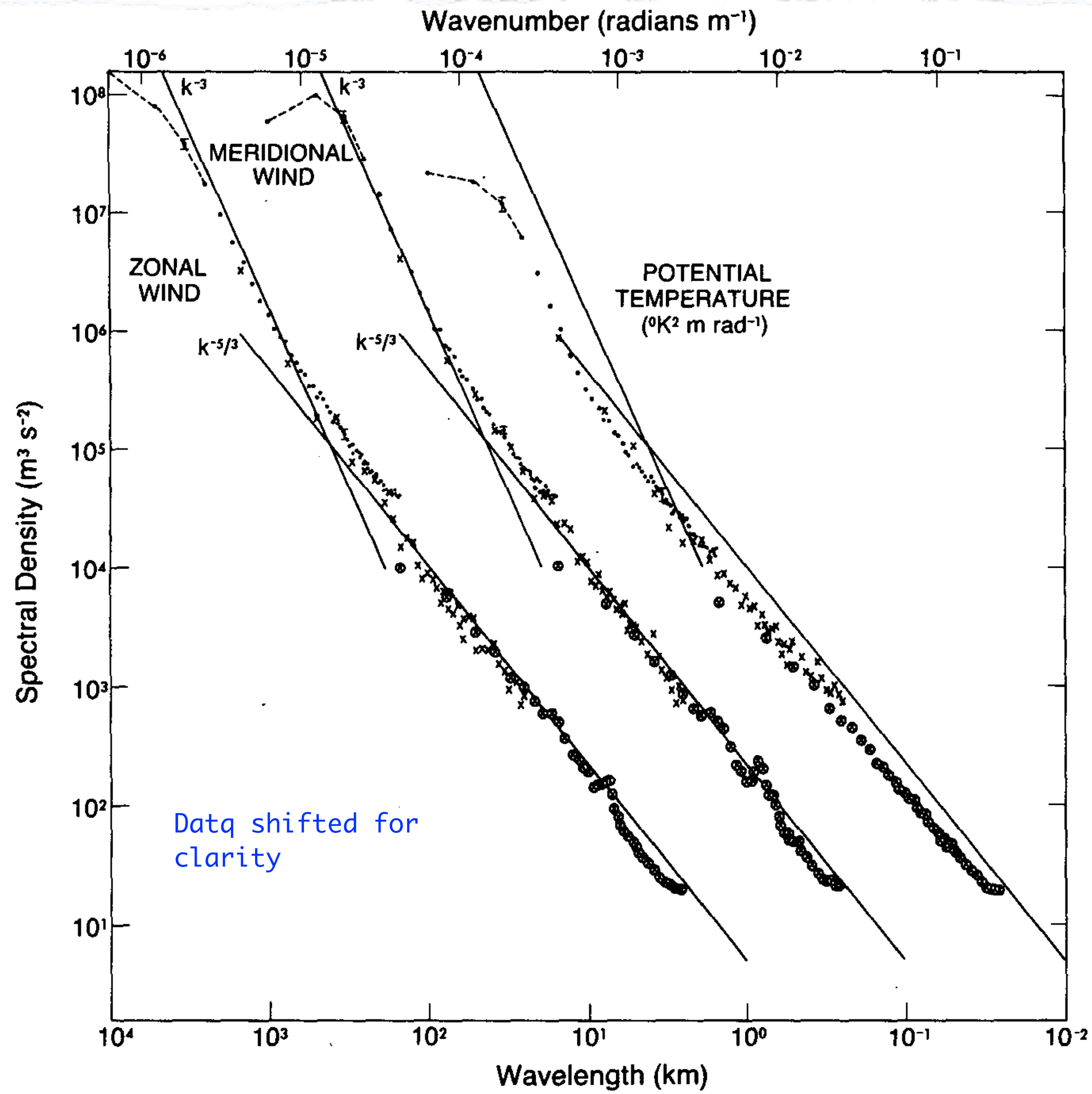


FIG. 3. Variance power spectra of wind and potential temperature near the tropopause from GASP aircraft data. The spectra for meridional wind and temperature are shifted one and two decades to the right, respectively; lines with slopes -3 and $-5/3$ are entered at the same relative coordinates for each variable for comparison.

Measurements of (u,v,b) along many flight tracks of commercial airliners (GASP 1970s) near tropopause 9-15km altitude

Apparent k^-3 scaling at large scales

Flatter k^-5/3 scaling at small scales

Transition at 500-700 km wavelength

More recent observations (MOZAIC 1990-2000) show the same

Large-scale spectrum compatible with down-scale enstrophy cascade of 2d or quasi-geostrophic turbulence, but physical origin of small-scale spectrum remained subject of debate.

Concluding comments

- Simple robust method for Helmholtz decomposition of horizontal velocity from one-dimensional data & linear wave-vortex decomposition.
- Useful for ship of flight tracks and for local analysis of observational/simulation data
- Hydrostatic wave energy spectrum for inertia-gravity waves computable solely based on observed horizontal velocities
- Strong assumptions of horizontal isotropy and vertical homogeneity
- Ongoing work on anisotropic version (with Max Kuang and Esteban Tabak) and on weakly nonlinear ageostrophic balanced flows using Omega equation (with Han Wang)