CORRESPONDENCE

Reply to "Comments on 'Revisiting the Relationship between Eyewall Contraction and Intensification'"

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ABSTRACT

In their comment, Kieu and Zhang critique the recent study of Stern et al. that examined the contraction of the radius of maximum wind (RMW) and its relationship to tropical cyclone intensification. Stern et al. derived a diagnostic expression for the rate of contraction and used this to show that while RMW contraction begins and accelerates as a result of an increasing negative radial gradient of tangential wind tendency inward of the RMW, contraction slows down and eventually ceases as a result of the increasing sharpness of the wind profile around the RMW during intensification. Kieu and Zhang claim that this kinematic framework does not yield useful understanding, that Stern et al. are mistaken in their favorable comparison of this framework to earlier work by Willoughby et al., and that Stern et al. are mistaken in their conclusion that an equation for the contraction of the RMW derived by Kieu is erroneous. This reply demonstrates that each of these claims by Kieu and Zhang is incorrect.

1. Introduction

Stern et al. (2015, hereafter S15) investigated the contraction of the hurricane eyewall and its relationship to the intensification of the maximum wind speed. It has long been understood that the eyewall and radius of maximum wind (RMW) tend to contract to a smaller size as the tropical cyclone (TC) intensifies. A widely accepted explanation for this phenomenon is the "convective ring theory" of Shapiro and Willoughby (1982, hereafter SW82) and Willoughby et al. (1982, hereafter W82), which hypothesizes that both contraction and intensification occur in response to sustained condensational heating in the eyewall updraft. Essentially, unbalanced eyewall heating drives a secondary circulation (radial and vertical motion), which acts to restore the TC toward thermal wind balance. Below and outside the region of heating, there is inflow, which through angular momentum advection increases the tangential winds. SW82 and W82 used a Sawyer–Eliassen equation with idealized vortices to diagnose the tangential wind tendency that results from heating near the RMW and showed that in general, the wind tendency tends to be greater inside of the RMW than at the RMW itself. Through this mechanism, the RMW will contract as the TC intensifies.

Stern (2010) and Stern and Nolan (2011) found that in idealized simulations of TCs using the Weather

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Research and Forecasting (WRF) Model, the RMW did contract during the initial period of intensification. However, in all cases, contraction slowed down and stopped while the storm was still intensifying. In fact, for a wide range of realistic initial conditions, most eyewall contraction occurred prior to most intensification, and in some cases, intensification continued for several days beyond the establishment of a quasi-steady size. Independently of Stern (2010) and Stern and Nolan (2011), Vigh (2010) found evidence from aircraft observations that in many TCs, contraction of the RMW tends to slow down or halt near the time when an eye forms and that contraction can be completed in the middle of the intensification process. S15 expanded upon the findings of Stern and Nolan (2011) and Vigh (2010), using both simulations and observations to systematically demonstrate this phenomenon.

S15 derived a diagnostic expression for the rate of change of the RMW. Utilizing this kinematic framework, S15 were able to explain why the eyewall might stop contracting in the middle of intensification. S15 also clarified a popular misconception regarding the RMW that the convective ring theory requires that intensification be accompanied by contraction. While SW82 and W82 did not themselves make this claim, and appeared to be aware that contraction and intensification need not be synonymous, this distinction became lost in many subsequent studies. Related to this idea that contraction is expected throughout the period of intensification, the RMW is often considered to maintain constant absolute angular momentum M as it contracts. Sometimes, it is even assumed that the RMW itself behaves as a material surface that can be advected. This latter assumption is not correct, and additionally, there is no intrinsic reason that M must remain constant in time at the RMW. As pointed out by S15, M surfaces can propagate through the RMW, and as long as the net tendency of M at the RMW is positive, intensification will continue, including at a constant or even an expanding RMW.

Independently of Stern and Nolan (2011) and Vigh (2010), Kieu (2012, hereafter K12) found evidence that eyewall contraction tends to stop within the middle of intensification, using an ensemble of WRF simulations [as described in Weng and Zhang (2012)] of Hurricane Katrina (2005), along with idealized simulations using the axisymmetric model of Rotunno and Emanuel (1987). K12 sought to explain this finding by deriving an equation for the rate of contraction of the RMW. Unfortunately, we believe that the model proposed by K12 is flawed. In their study, S15 briefly discussed these shortcomings in K12 and explained that although K12 did contribute to our understanding of eyewall contraction by providing additional evidence that intensification can occur at a fixed RMW, the framework used by K12 to explain this phenomenon is invalid.

In their comment on S15, Kieu and Zhang (2017, hereafter KZ17), assert three primary claims: (i) that the diagnostic contraction equation of S15 does not provide useful insight into the processes of contraction and intensification; (ii) that the statement by S15 that the equation of W82 is an approximation to the equation of S15 is incorrect; and (iii) that S15's critique of K12 is mistaken and that the equation of K12 is indeed mathematically correct and physically valid for explaining the contraction of the RMW.

In our reply here, we will demonstrate that all three claims of KZ17 are incorrect. In section 2, we review the contraction equation of S15 and elaborate on the insight that the resulting analysis provides into our understanding of the relationship between eyewall contraction and TC intensification. In section 3, we compare this equation to that of W82 and demonstrate that the equation of W82 is equivalent to a one-sided finite-difference approximation to the equation of S15, which is continuous and exact. In section 4, we review the derivation of K12 and show that it is erroneous and that it leads to an incorrect physical interpretation of the dynamics governing contraction. In sections 5 and 6, we present a discussion and summary, respectively.

2. A kinematic understanding of eyewall contraction

A key insight of S15 is that although a negative radial gradient of tangential wind tendency at the RMW corresponds to a contraction of the RMW, the rate of contraction is dependent not just on this gradient but also on the degree of curvature of the radial profile of tangential wind. All else being equal, it is much easier for a broad maximum to contract than it is for a sharply peaked maximum. The fact that the sharpness of the wind profile affects the rate of contraction is not a new discovery, and in discussing their convective ring theory, SW82 (p. 393) wrote, "In the absence of other physical processes, the radial profile of v would develop a sharp peak near the maximum of $\partial v/\partial t$. The continued inward movement of the RMW depends on processes that maintain both the heat and momentum sources and the rounded profile of v near the RMW." As far as we are aware, no study prior to S15 quantified or even examined the relative influence of the sharpness of the tangential wind profile on the contraction of the eyewall. Few if any studies subsequent to SW82 even referred to this influence, and the idea that contraction is only a function of the radial gradient of wind tendency became the conventional wisdom.

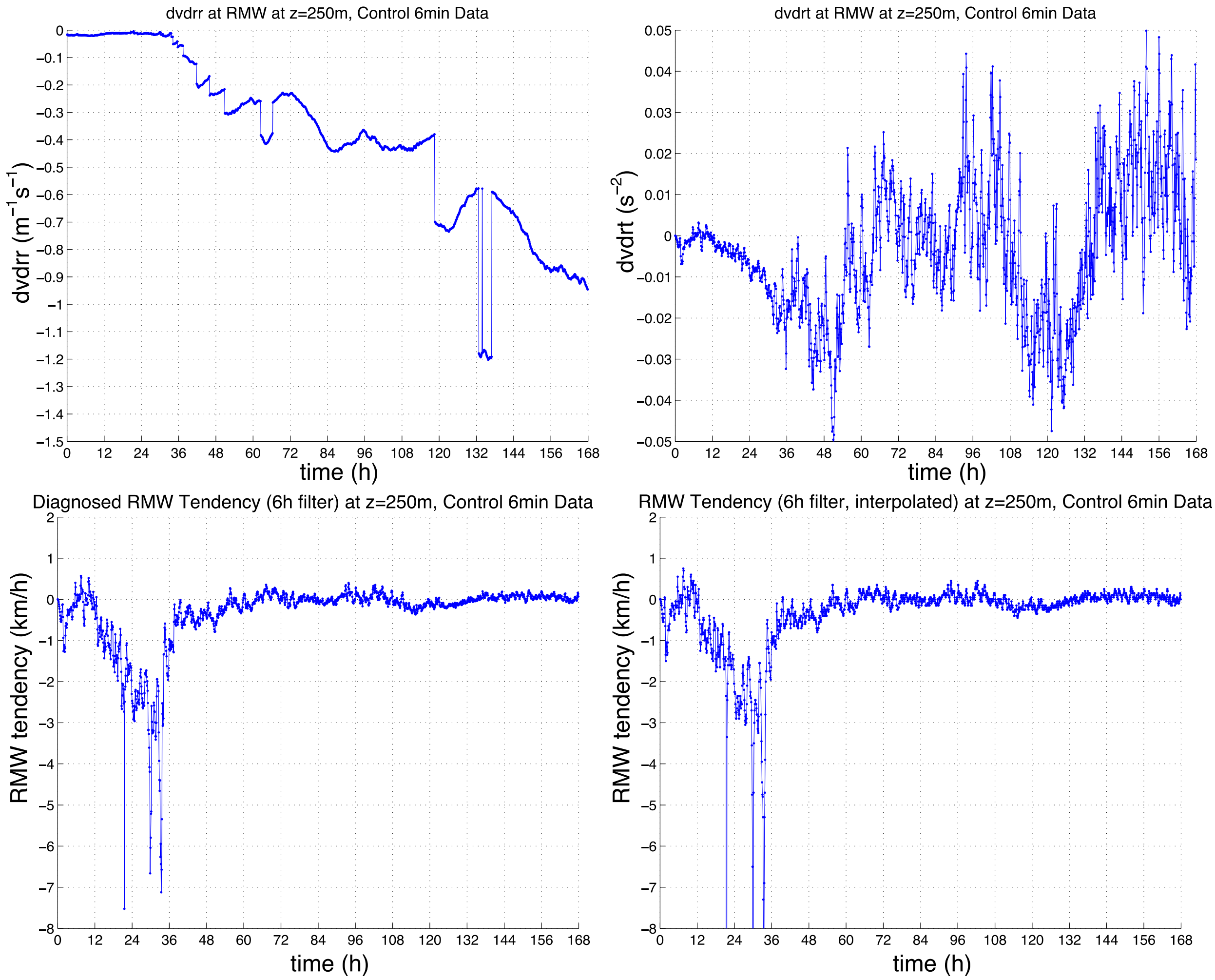

FIG. 1. Figure 6 from S15. Time series of (top left) $\partial^2 V/\partial r^2$, (top right) $(\partial/\partial r)(\partial V/\partial t)$, (bottom left) the diagnosed time tendency of the RMW from (1), and (bottom right) the actual tendency of the RMW. The time series for the top panels have been multiplied by 10^{-6}.

S15 derived the following diagnostic expression for the rate of change of the RMW:

$$\frac{d\mathrm{RMW}}{dt} = -\left.\frac{(\partial/\partial r)(\partial V/\partial t)}{\partial^2 V/\partial r^2}\right|_{\mathrm{RMW}}, \qquad (1)$$

where V is the tangential wind speed. As noted in S15, this equation is only valid when evaluated at the RMW. As KZ17 do not dispute the validity of this equation, we refer the reader to S15 for the derivation of (1), which is (7) of S15. The numerator of (1) is the radial gradient of the time tendency of tangential wind at the RMW, and the denominator is the curvature of the radial profile of tangential wind, also at the RMW. The curvature at the location of maximum wind is negative definite, so the sign of the numerator determines whether there is contraction or expansion of the RMW. It is clear that both terms contribute to the magnitude of the RMW tendency, but the relative contributions are not intuitively obvious.

S15 examined time series of the rate of change of the RMW diagnosed by (1), the contributions of the radial gradient and curvature terms, and the actual rate of change of the RMW. In S15, this is Fig. 6, which we reproduce here as Fig. 1. There is excellent agreement between the diagnosed and actual RMW tendency, demonstrating the validity of (1). In this control simulation, contraction begins at around $t = 12$ h, and the rate of contraction increases to 2–3 km h^{-1} by $t = 30$ h. This acceleration in the contraction rate is entirely due to the increasingly negative radial gradient of tangential wind tendency at the RMW. Shortly before $t = 36$ h, the contraction begins to rapidly decelerate, and the RMW becomes quasi steady by about $t = 48$ h. It might be expected that this deceleration and halting of contraction is associated with a decrease in the magnitude of the radial gradient of wind tendency, that is, that the region of peak wind tendency is becoming closer to the RMW, eventually becoming collocated. Indeed, this had been suggested by the study of Schubert and Hack (1982, p. 1693), who wrote, "One can picture the tendency for the radius of maximum wind to move inward until in coincides with the inner edge of the heated region." However, S15 found that in their simulation, this was not the case. Instead, as can be seen in Fig. 1, the slowdown of contraction is due to the dramatic increase in the sharpness of the tangential wind profile, as $\partial^2 V/\partial r^2$ increases by a factor of 10 over only 6 h. In this same period, $(\partial/\partial r)(\partial V/\partial t)$ is actually still increasing in magnitude, which would otherwise result in a continuation of the accelerating contraction. For the remainder of the simulation beyond $t = 48$ h, there are alternating periods where the radial gradient is strongly positive or negative. However, there is little substantial movement of the RMW in either direction, and this is because of the increasing curvature of the radial profile of tangential winds. This phenomenon can be visualized more clearly by looking at the evolution of radial profiles of tangential wind, and to show this, we reproduce Fig. 7 of S15 as Fig. 2 here. It can be seen that the RMW of an initially broad profile rapidly contracts and that contraction slows down and stops as the profile sharpens.

In their comment, KZ17 argue that the analysis of S15 "provides little understanding of the processes underlying the RMW contraction." We believe that this criticism is unjustified. KZ17's concern here is based on the fact that (1) is diagnostic and kinematic and not prognostic and dynamical. The point that (1) is purely diagnostic was made clear in S15. Because (1) is diagnostic, we can only partition the relative contributions to contraction after already knowing the evolution of the wind field. This is also clear from the text of S15. Additionally, although our contraction equation is diagnostic, there are clear benefits to this approach. For example, there are no assumptions made at all in deriving (1), and this equation is valid for any local maximum in any radial profile of tangential wind. Further, an analogous equation could be derived for the maximum or minimum in any scalar field, for example, the height of the temperature inversion within the hurricane eye. It seems that KZ17 recognize this and intend to criticize S15 on the grounds that (1) cannot predict its own component terms.

S15 acknowledged that (1) on its own does not provide a dynamical framework for understanding contraction. Ultimately, we would like to understand physically why contraction slows down and stops when it does and how this process relates to the continued intensification of the hurricane. To understand the dynamics, we first must realize that the sharpness of the tangential wind profile matters for the continued contraction of the RMW. And

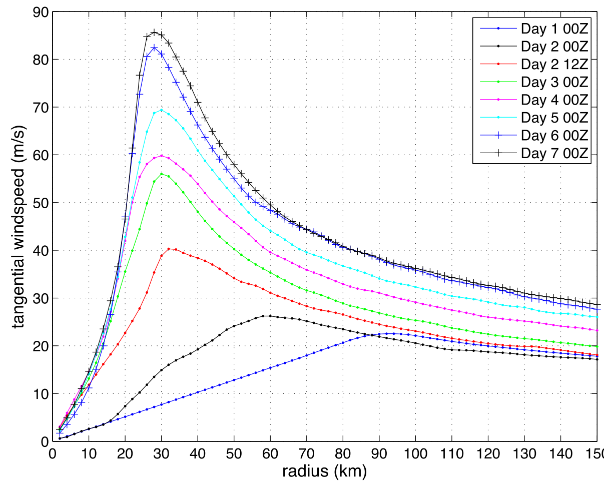

FIG. 2. Figure 7 from S15. Radial profiles of azimuthal-mean tangential wind for the control simulation at 0000 UTC each day from day 1 to day 7. To illustrate the period of rapid contraction, 1200 UTC day 2 is also plotted.

from this, we may know that it is important to investigate the physical processes that contribute to the tendency for the RMW to become more peaked as the TC intensifies. The kinematic framework of S15 is also useful because any dynamical theory for contraction of the RMW must be consistent with (1). As we will explain in section 5, the model of K12 is inconsistent with (1) and therefore cannot be correct.

We also note that in their discussion of S15, KZ17 state that one key conclusion of S15 is that RMW contraction is due to the radial gradient of wind tendency and not to the curvature term. Although we did find that during the rapid contraction phase, the changes in the radial gradient of wind tendency were dominant, our primary focus actually concerned the subsequent rapid slowdown of contraction. This rapid slowing of contraction is associated with increases in the curvature of the wind profile at the RMW, and we wish to clarify that this finding was the key conclusion of S15.

KZ17 state that if we know the wind field and its tendency, then the RMW and its contraction rate are also already known, without a need for (1). This is true. However, by using their diagnostic equation, S15 were able to determine that the RMW stopped contracting because of the rapid increase in the curvature of the tangential wind profile and not because of a change in the magnitude or location of wind tendency.

3. The similarity of the equations of W82 and S15

In S15, we mentioned that W82 had previously presented an equation for the rate of contraction of the RMW, and we noted that their equation is essentially a finite-difference approximation to (1). KZ17 dispute this claim of S15 and state that (1) "has no clear similarity to the RMW contraction equation presented in Willoughby et al. (1982)." Here, we show that the contraction equation of W82 is indeed similar to and can be derived from (1).

In the interest of brevity, S15 did not explicitly show how (3) of W82 related to (7) of S15, and we admit that it may not be readily apparent from a cursory comparison of the two equations. For completeness, (3) of W82, using the authors' original notation, is

$$c \equiv \frac{dr_{\max}}{dt} = \frac{(\partial v/\partial t)_{\max} - (dv/dt)_{r_{\max}}}{\partial v/\partial r}, \qquad (2)$$

where c and $dr_{\max}/dt$ are equivalent expressions for the rate of change in time of the RMW, $(\partial v/\partial t)_{\max}$ is the maximum local time tendency of tangential wind (not necessarily at the RMW), $(dv/dt)_{r_{\max}}$ is the tangential wind tendency at the RMW, and $\partial v/\partial r$ is the "radial shear inside the eyewall," which W82 then approximate as $\partial v/\partial r \approx v_{\max}/r_{\max}$. From this approximation, the fact that W82 state these terms are computed over a finite region corresponding to "the width of the prominent wind tendency peak," and the fact that all tendencies are estimated statistically over a finite time interval corresponding to the observational sampling period, we can infer that (2) is effectively a one-sided finite-difference approximation to the actual RMW tendency. KZ17 criticize our use of this particular finite-difference approximation as subjective and arbitrary, but it was W82 who originally made this approximation, so our choice here is not arbitrary. We are merely demonstrating that by making such an approximation (as we can infer that they did), one can arrive at their diagnostic contraction equation starting from (1). It is true that by making different approximations, we could arrive at different final equations, but this is not relevant to the comparison of our equation to that of W82.

For clarity of comparison, S15 rewrote (2) using the same notation as (1), which we restate here:

$$\frac{d\mathrm{RMW}}{dt} = \frac{(\partial V/\partial t)_{\max} - (\partial V/\partial t)_{\mathrm{RMW}}}{\partial V/\partial r}. \qquad (3)$$

KZ17 note that (3) differs from (2) in that the partial derivative $(\partial V/\partial t)_{\mathrm{RMW}}$ has replaced the total derivative $(dV/dt)_{\mathrm{RMW}}$ (using the terminology of S15). KZ17 assert that this represents an error by S15. Actually, S15 is correct here, because at the RMW itself, $\partial V/\partial t$ and dV/dt are mathematically equivalent. The total derivative following the RMW can be expanded as

$$\left.\frac{dV}{dt}\right|_{\mathrm{RMW}} = \left.\frac{\partial V}{\partial t}\right|_{\mathrm{RMW}} + \frac{d\mathrm{RMW}}{dt}\left.\frac{\partial V}{\partial r}\right|_{\mathrm{RMW}} = \left.\frac{\partial V}{\partial t}\right|_{\mathrm{RMW}}. \qquad (4)$$

In other words, the time tendency of the maximum winds and the local time tendency of winds at the location of the maximum are the same, because the radial gradient of tangential wind $\partial V/\partial r$ is by definition zero at the RMW. Note that here (and in W82), the total derivative is defined as following the RMW and not the traditional meaning as following a Lagrangian parcel. The Lagrangian parcel derivative is *not* equivalent to the rate of change of the maximum. But since the rate of change in the maximum tangential wind speed is always equal to the local Eulerian rate of change of the tangential wind speed at the RMW, we may substitute these terms for each other, so S15 were not mistaken in their statement and evaluation of (3) of W82. KZ17 argue that when expressed as finite differences in time, these terms are no longer equivalent, because the RMW varies in time. However, S15 express the time derivatives in their continuous form, as do W82 in their (3), and so KZ17's criticism is not relevant. W82 *evaluate* their contraction equation using the rate of change of the maximum winds over a finite time interval, which is a further approximation of their (3), required for their analysis of observational data. But this does not change the fact that their (3) as written is indeed equivalent to (3) herein. For simplicity, hereafter, we refer exclusively to (3) as the contraction equation of W82.

KZ17 point out that (3) has no second derivatives, unlike (1), and they state that they are unable to find any way to derive (3) from (1) [respectively, (8) and (7) of S15). Here, we show such a derivation. First, we approximate the outer radial derivatives of (1) as finite differences:

$$\frac{\partial}{\partial r}\frac{\partial V}{\partial t} \approx \frac{(\partial V/\partial t)_{\mathrm{RMW}} - (\partial V/\partial t)_{\max}}{\Delta r}$$

and

$$\frac{\partial^2 V}{\partial r^2} \approx \frac{(\partial V/\partial r)_{\mathrm{RMW}} - (\partial V/\partial r)_{\max}}{\Delta r},$$

where here, we adopt the notation of W82 to indicate that $(\partial V/\partial t)_{\mathrm{RMW}}$ is valid at the RMW and $(\partial V/\partial t)_{\max}$ is valid at the location of the maximum time tendency. Therefore, (1) becomes

$$\frac{d\mathrm{RMW}}{dt} = -\frac{\dfrac{(\partial V/\partial t)_{\mathrm{RMW}} - (\partial V/\partial t)_{\max}}{\Delta r}}{\dfrac{(\partial V/\partial r)_{\mathrm{RMW}} - (\partial V/\partial r)_{\max}}{\Delta r}}. \qquad (5)$$

Next, we eliminate Δr because it is present in both the numerator and denominator, and we absorb the negative sign into the numerator to get

$$\frac{d\mathrm{RMW}}{dt} = \frac{(\partial V/\partial t)_{\max} - (\partial V/\partial t)_{\mathrm{RMW}}}{(\partial V/\partial r)_{\mathrm{RMW}} - (\partial V/\partial r)_{\max}}. \tag{6}$$

We then can eliminate the first term in the denominator of (6) because $\partial V/\partial r$ is by definition zero at the RMW, which results in

$$\frac{d\mathrm{RMW}}{dt} = \frac{(\partial V/\partial t)_{\max} - (\partial V/\partial t)_{\mathrm{RMW}}}{-(\partial V/\partial r)_{\max}}. \tag{7}$$

Finally, we recognize that (3) of W82 is actually defined such that contraction is positive, so we must remove the negative sign from the denominator of (7) to transform our formulation to be consistent with W82's convention. At this point, we have arrived at (3), which, as we explained above, is equivalent to (2), the original (3) of W82.

KZ17 attempt to derive (3) of W82, but their derivation appears to be in error. Their (4) is based on the total derivative of tangential wind following some point, which they then take to be the RMW. For a single dimension (KZ17 write their equation in three dimensions but only utilize the radial dimension), this would give

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{dR}{dt}\frac{\partial v}{\partial r}. \tag{8}$$

KZ17 rearrange (8) to solve for dR/dt and then express this as a finite difference. However, at the RMW itself, $\partial v/\partial r = 0$, dv/dt following the RMW is equal to $\partial v/\partial t$ at the RMW, and dR/dt, the rate of change of the RMW, drops out of the equation and so cannot be directly solved for in this manner. One could perhaps argue that by approximating derivatives as finite differences in radius, this problem of dR/dt dropping out can be bypassed. However, since the analytical form of (8) derived at the RMW renders a solution for dR/dt impossible, we believe that it is not proper to arrive at (3) of W82 in this manner. Equation (3) of W82 can only be derived by starting from the analytical form of (1) and approximating derivatives as one-sided finite differences, as we showed above. In any case, an alternative derivation of (3) of W82 would in no way invalidate the fact that this equation can indeed be derived from (1).

KZ17 refer to $(\partial v/\partial t)_{\max}$ as the "maximum tendency of the tangential wind at a fixed radius (e.g., at the RMW at the next time step)." However, this assumes that the future location of the RMW will be at the location of the largest local tendency, which is not necessarily the case. Additionally, although KZ17's schematic in their Fig. 1 might illustrate a potential observational method for diagnosing contraction, it does not actually represent what W82 did in their analysis. W82 used all legs of data from a single flight to statistically estimate the time tendency of winds at fixed radii and then used the estimate of the radial profile valid at the central time of the flight for their calculations (see Fig. 14 of W82). So although W82 used data from an entire flight, they actually applied their equation to a single time and a single RMW location. Therefore, the claim made by KZ17 that the derivatives in the equations of W82 and S15 "represent two totally different concepts" is incorrect.

Finally, we note that we have examined the textbook of Petterssen (1956), which is cited by W82 (and KZ17), in order to better understand the origin of the contraction equation of W82. Petterssen (1956, 44–56) describes the kinematics of the pressure field and mathematically defines the axis of a trough as satisfying $\partial p/\partial x = 0$ and $\partial^2 p/\partial x^2 > 0$, where p is pressure and x is the horizontal direction normal to the trough. Petterssen then defines a differential operator [his (3.3.2)] for a moving system of coordinates as

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + C\frac{\partial}{\partial x}, \tag{9}$$

where C is the translation speed of the coordinate system, $\partial/\partial t$ is the conventional partial time derivative, and $\delta/\delta t$ is the time derivative at the same point in the moving coordinate system. Applying this operator to the definition of a trough yields

$$\frac{\delta}{\delta t}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial x}\right) + C\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right). \tag{10}$$

Since $\partial p/\partial x = 0$ at the trough axis, then the time derivative of $\partial p/\partial x$ following the moving trough is also zero, and the lhs of (10) vanishes, allowing for the solution of the propagation speed of the trough C as

$$C = -\frac{\partial^2 p/\partial x\,\partial t}{\partial^2 p/\partial x^2}. \tag{11}$$

Equation (11) is (3.4.4) of Petterssen (1956). It is apparent that this is precisely analogous to (1), which is (7) of S15, except with a pressure minimum instead of a wind maximum and using x instead of r. An analogous equation defines (in a single dimension) the continuous propagation speed of a local maximum or minimum of any scalar field. As shown above, the equation of W82 can be derived as a finite-difference approximation to (1), and since (11) is analogous to (1), it is now clear how W82 adapted the technique of Petterssen (1956) to arrive at their equation for the contraction of the RMW.

4. The contraction equation of K12

After demonstrating that the RMW stopped contracting during the middle of intensification for simulations of Hurricane Katrina as well as for idealized axisymmetric simulations, K12 attempted to derive a dynamical equation that governs the rate of change of the RMW. S15 briefly discussed this theoretical framework of K12 and explained that it was conceptually flawed. Additionally, S15 pointed out that there was an important error in the derivation of K12's equation, such that it was actually impossible to correctly arrive at a prognostic dynamical equation for the contraction of the RMW. Therefore, S15 argued that the model of K12 is not valid for understanding contraction. In their comment, KZ17 argue that S15 were mistaken in their assessment of K12, that the equation of K12 is indeed correct, and that the theory of K12 does indeed help explain contraction. Here, we point out the key error in the derivation of K12 and explain in detail how this results in a final equation that is incorrect and cannot be corrected.

The steps used by K12 and KZ17 to demonstrate their derivation are not identical. For simplicity, here we will mainly refer to the form of the derivation in KZ17. KZ17 start by assuming a vortex with a particular radial profile of tangential wind, whereby there is always solid-body rotation at and inward of the RMW, such that in this region

$$v(r,t) = \Omega(t)r, \tag{12}$$

where Ω is the angular velocity, which is assumed to be only a function of time. KZ17 then substituted this assumed profile into an axisymmetric version of the tangential momentum equation [their (8)] defined as follows:

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial r} - \frac{uv}{r} - w\frac{\partial v}{\partial z} - fu - \frac{C_D}{H}v\sqrt{(u^2+v^2)}, \tag{13}$$

where u and w are the radial and vertical velocities, f is the Coriolis parameter, C_D is the drag coefficient, and H is a constant boundary layer depth. Following the substitution of (12) into (13), KZ17 arrive at

$$\frac{d\Omega}{dt}r = -u\frac{\partial v}{\partial r} - (\Omega + f)u - \frac{C_D}{H}v\sqrt{(u^2+v^2)}; \tag{14}$$

here, KZ17 eliminate the vertical advection term by assuming a barotropic vortex where $\partial v/\partial z = 0$, and they combine uv/r and fu into a single term $(\Omega + f)u$ based on the definition of Ω. Crucially, KZ17 replace $\partial v/\partial t$ on the lhs of (13) with $(d\Omega/dt)r$ on the lhs of (14), and this is incorrect; $\partial v/\partial t = (\partial\Omega/\partial t)r$, but the total derivative $d\Omega/dt$ is *not* equal to the partial derivative $\partial\Omega/\partial t$, so these derivatives are not interchangeable. KZ17 claim that because they are assuming that Ω is a function of time alone at and inward of the RMW, the partial and total derivatives are indeed interchangeable. They are mistaken, however, because such an assumption can never be valid at the RMW itself. The total derivative of Ω following the RMW can be expanded as

$$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + \frac{d\mathrm{RMW}}{dt}\frac{\partial\Omega}{\partial r}. \tag{15}$$

KZ17 are assuming that because they have specified that Ω is constant at and inward of the RMW, then $\partial\Omega/\partial r = 0$ at the RMW itself. This is incorrect. Technically, $\partial\Omega/\partial r$ is undefined at the RMW, because this derivative is discontinuous for a Rankine vortex. KZ17 imply that an equivalent solution could be found for a smooth and continuously differentiable profile, and they refer to the appendix of K12 as an example of such a profile. However, K12 do not show that such a profile allows for a solution for the rate of change of the RMW. Moreover, the appendix of K12 does not actually present a specific function for this smooth profile but instead simply refers to a generic smooth function $f(r)$ that modifies the Rankine vortex near the RMW. In a footnote, KZ17 give an example of a smooth profile and assert that it satisfies the condition that Ω is constant up to and including the RMW. This assertion cannot be true, and elsewhere, the authors acknowledge this fact, arguing instead that Ω is approximately constant. Contrary to KZ17's assumption here, any smooth profile could no longer have Ω constant up to the RMW, since the deviation from a linear profile of v results in radially varying Ω, by definition.

Irrespective of the problem of the undefined radial derivative for the assumed Rankine profile, we know that $\partial\Omega/\partial r$ cannot be zero at the RMW for any radial profile of tangential wind, since

$$\frac{\partial\Omega}{\partial r} = \frac{\partial(v/r)}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}. \tag{16}$$

As $\partial v/\partial r$ is zero at the RMW by definition, then $\partial\Omega/\partial r$ by definition can never be zero at the RMW. Therefore, the total derivative $d\Omega/dt$ following the RMW can never be equal to the partial derivative $\partial\Omega/\partial t$. There is no situation under which these derivatives are interchangeable, so (14) must be incorrect. Because (14) is incorrect, all other equations in KZ17 that follow from (14) are also incorrect.

In the next step of their derivation, K17 assume that u at the RMW is always negative and then evaluate (14) at the RMW to get

$$\frac{d\Omega}{dt}R = (\Omega + f)|U| - \frac{C_D}{H}V\sqrt{(U^2+V^2)}, \tag{17}$$

where R is the RMW and all variables are evaluated at the RMW. This is incorrect, again because the lhs should be $(\partial\Omega/\partial t)R$, not $(d\Omega/dt)R$. Next, KZ17 use the definition of angular velocity to write

$$\frac{dV}{dt} = \frac{d\Omega}{dt}R + \Omega\frac{dR}{dt}. \tag{18}$$

KZ17 rearrange (18) to solve for $(d\Omega/dt)R$ and substitute this into (17) to get

$$\Omega\frac{dR}{dt} = \frac{dV}{dt} - (\Omega + f)|U| + \frac{C_D}{H}V\sqrt{(U^2+V^2)}. \tag{19}$$

Equation (19) is incorrect because $(d\Omega/dt)R$ cannot actually be eliminated with this substitution, as this term is incorrectly present on the lhs of (17).

In the final step of their derivation, KZ17 divide (19) by Ω, assume that $\Omega \gg f$ and that $V \gg U$, and rearrange to arrive at their equation for the rate of change of the RMW:

$$\frac{dR}{dt} = -|U| + \frac{C_D}{H}VR + \frac{R}{V}\frac{dV}{dt}. \tag{20}$$

Equation (20) is incorrect because (19) is incorrect, so the end result of the derivation of K12 and KZ17 is invalid.

Finally, we can show that if the error in substituting $d\Omega/dt$ for $\partial\Omega/\partial t$ is not made, then an equation for the rate of change of the RMW cannot be derived. A correct version of (17) would be

$$\frac{\partial\Omega}{\partial t}R = (\Omega + f)|U| - \frac{C_D}{H}V\sqrt{(U^2+V^2)}. \tag{21}$$

Using (15) to solve for $\partial\Omega/\partial t$, we can substitute for $\partial\Omega/\partial t$ in (21) to get

$$\frac{d\Omega}{dt}R - \frac{dR}{dt}\frac{\partial\Omega}{\partial r}R = (\Omega + f)|U| - \frac{C_D}{H}V\sqrt{(U^2+V^2)}. \tag{22}$$

Now we can use (18) to solve for $(d\Omega/dt)R$, as is done by KZ17, and substitute for $(d\Omega/dt)R$ in (22) and then rearrange to get

$$\Omega\frac{dR}{dt} + \frac{dR}{dt}\frac{\partial\Omega}{\partial r}R = \frac{dV}{dt} - (\Omega + f)|U| + \frac{C_D}{H}V\sqrt{(U^2+V^2)}. \tag{23}$$

At the RMW, $\partial\Omega/\partial r = -v/r^2$ by definition, and so the lhs of (23) can be written as $(dR/dt)[\Omega - (V/R^2)R] = (dR/dt)(\Omega - \Omega) = 0$, which is identically zero. Thus, the resulting equation no longer references the RMW at all.

KZ17 claim that we have introduced a new assumption in our criticism of their derivation. This is not the case. Instead, we have only shown that the assumption by KZ17 that Ω is constant in radius at the RMW is untenable, as it can never be true for any wind profile. Effectively, KZ17 make two mutually contradictory assumptions: that there is a local maximum in the tangential wind speed and that the angular velocity has no spatial variation near this maximum. KZ17 argue that it is not appropriate for us to refer to their derivation as erroneous, because "different assumptions lead to different equations." But what we have shown is that no kinematically consistent profile can lead to the contraction equation that the authors derive. Therefore, it is appropriate to consider this derivation to be in error.

KZ17 raise the point that our explanation of the error in K12 and KZ17 differs from what we wrote in S15 and conclude that this difference is evidence that the equations of K12 and KZ17 are actually correct. It is true that S15 differs from the above explanation, but this in no way alters the fundamental reason that K12 and KZ17 are incorrect: that they inappropriately treat the partial and total derivatives of Ω as equivalent at the RMW. In S15, we stated that K12 substituted their (2) into their (4) to arrive at their (5) but that they left out a term $\Omega(dR/dt)$, which allowed them to inappropriately eliminate $R(d\Omega/dt)$ and obtain a solution for dR/dt. The derivation in K12 is not as clear or as complete as that presented in KZ17, so in S15, we inferred that leaving out a term was the only way that K12 could have arrived at their final equation. In particular, nowhere in K12 do the authors state that they are directly substituting $d\Omega/dt$ for $\partial\Omega/\partial t$ or that they assumed that these terms are equivalent. In retrospect, the derivation in KZ17 makes it evident that K12 were making this substitution and assumption, and in this respect, we were mistaken in S15 that the error in K12 stemmed from accidentally neglecting a term in their derivation. The error that S15 thought K12 had made and the error that we have shown above that K12 and KZ17 actually made have an equivalent result: they both inappropriately allow for a solution of dR/dt.

5. Discussion

In the absence of friction, the radial winds advect momentum surfaces, and the tangential winds increase anywhere that M surfaces have been brought inwards. If there is inflow over some radial range that includes the RMW, then the tangential winds are increasing everywhere in this region. However, this information on its own tells us nothing about the propagation of the RMW. The RMW might be contracting, but it might be constant or even expanding. As shown in S15 and reiterated here, it is the sign of the gradient of the tangential wind

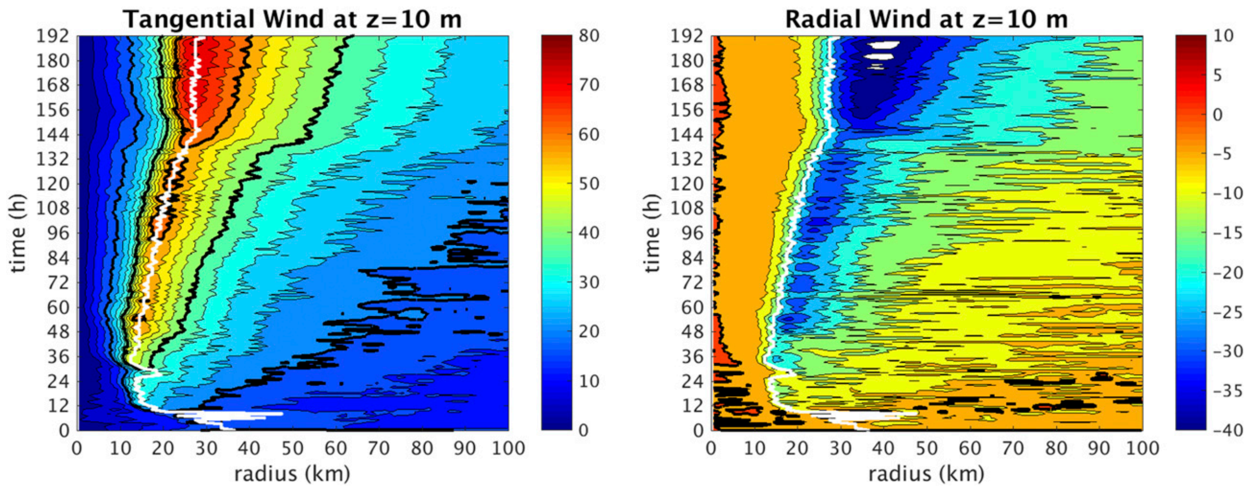

FIG. 3. Radius–time Hovmöller plots of (left) tangential and (right) radial wind at 10-m height for an axisymmetric simulation. The RMW is indicated by the white line. For tangential winds, the contour interval is $5\,\mathrm{m\,s^{-1}}$, with thick lines every $20\,\mathrm{m\,s^{-1}}$. For radial winds, the contour interval is $5\,\mathrm{m\,s^{-1}}$, and the zero contour is thickened.

tendency that determines the direction of RMW propagation, not the sign of the wind tendency itself. The sign of the radial velocity determines the sign of the tangential wind tendency (in the absence of other forces), but it does not determine the sign of the gradient of the tendency.

The contraction equation of K12 and KZ17 [here, (20)] is only dependent on local variables, and there are no radial derivatives. But we know that kinematically, the contraction of the RMW is dependent on the radial gradient of the wind tendency; therefore, any dynamical theory of contraction must contain information on the spatial variation of the flow. Kinematically, if the tangential winds are increasing more rapidly inward of the RMW than they are outward of the RMW, then the RMW must be contracting. But (20) has no reference to any radial variation.

As with the radial velocity, K12 and KZ17 treat surface friction as having a direct effect on the RMW, and this is physically incorrect. Friction results in a (generally negative) tangential wind tendency, but again, this tendency itself is not directly related to the propagation of the RMW; it is the gradient of the frictional tendency on tangential winds that matters. If, for example, the magnitude of the frictional spindown were maximized outward of the RMW, then the RMW would tend to contract. In (20), the friction term is positive definite. In other words, the theory of K12 and KZ17 predicts that the direct effect of friction can only act to expand the RMW (or oppose contraction). Additionally, KZ17 interpret the effects of friction in an inconsistent manner. Despite the fact that their equation predicts that friction always opposes contraction, they write that "an immediate consequence of" their equation is that "the RMW contraction rate will be faster for larger drag coefficient C_d." Since changing C_d will alter both drag and inflow, it is not actually possible to assess the net sensitivity to C_d in this manner; nevertheless, it is clear that the direct effect of increasing C_d in the context of their equation would be to render contraction slower, not faster.

As noted above, the sign of the radial velocity at the RMW is itself not directly relevant for determining the propagation of the RMW. KZ17 argue that "it is not likely to have a situation in which the inflow can lead to an expansion of the RMW." However, the likelihood of this occurring is not relevant to the validity of the theory of K12; all that matters is whether this situation is possible. The theory of K12 predicts that inflow at the RMW can only be associated with a tendency to contract the RMW, yet it is clearly possible for the RMW to expand within inflow. This can be seen, for example, by comparing Figs. 8, 12, and 15 of Wang (2009). To more clearly illustrate that expansion of the RMW can quite easily occur within inflow, in Fig. 3, we show radius–time Hovmöller diagrams of tangential and radial wind for an example axisymmetric simulation using CM1 (Bryan and Fritsch 2002). For this simulation, we use a similar setup to "Setup B"/"Res2" of Bryan (2012), except SST = 28°C, and the drag coefficient C_d is wind speed dependent. We use a modified Rankine vortex with initial RMW = 36 km and a decay coefficient of 0.25, which results in rapid contraction early in the simulation. It can clearly be seen that the radial winds at and in the vicinity of the RMW are continuously negative, yet after about 36 h, the RMW expands outward. Note that

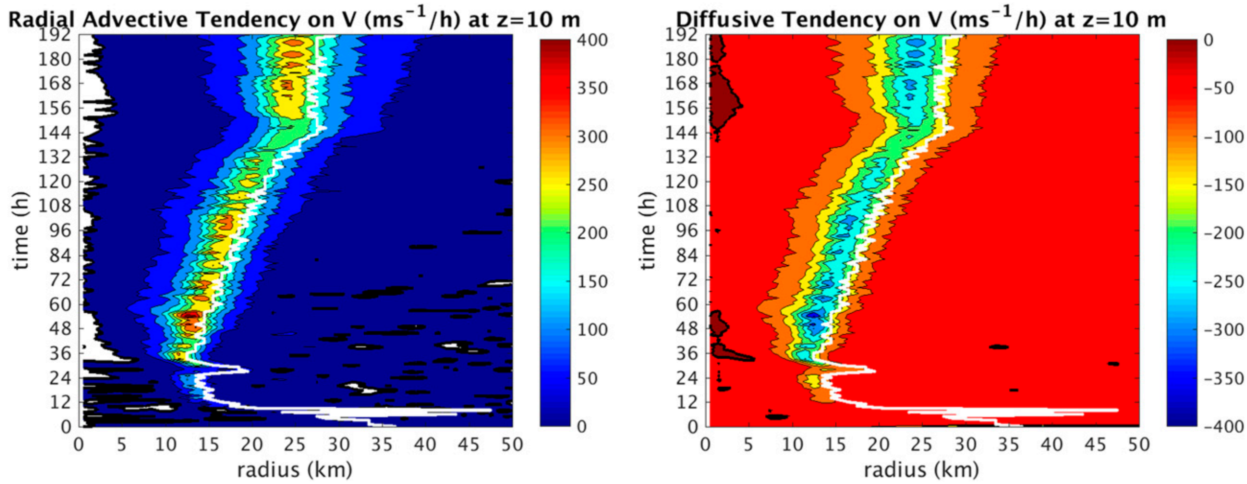

FIG. 4. Radius–time Hovmöller plots of the tendency on tangential winds at 10-m height, from (left) radial advection and (right) the sum of radial and vertical diffusion, for the same simulation as in Fig. 3. The RMW is indicated by the white line. The contour interval is 50 m s^{-1} h^{-1}, and the zero contour is thickened. Note that the radial range shown is different from that in Fig. 3.

this does not imply that the RMW expands because of the inflow; we are merely demonstrating that expansion can occur in the presence of inflow. KZ17 support their claim that expansion within inflow is unlikely by showing that for a particular assumed radial profile of tangential wind [their (15) and (16)], the tangential wind tendency resulting from a given value of radial inflow is greater inside the RMW than outside the RMW. However, this alone tells us little about the likelihood of expansion, because the actual radial profile of tangential wind tendency is highly dependent on the radial profile of radial velocity, which is unknown in their analysis.

Although the signs of the wind tendencies associated with friction and radial advection do not determine the tendency of the RMW, TC boundary layer dynamics do tend to constrain the spatial structure of the wind tendencies, such that the signs of the tendencies are correlated with the sign of their radial gradients. Figure 4 shows radius–time Hovmöller plots of the tangential wind tendency from radial advection of angular momentum and the combined tendency from vertical and horizontal diffusion for the simulation shown in Fig. 3. At the RMW, there is generally inflow, and the radial distributions of M and u are such that there is also generally a negative radial gradient of the associated tangential wind tendency. The diffusive tendency is generally negative at the RMW, but with the peak inward of the RMW, so there is generally a positive radial gradient of this tendency at the RMW. Because the tendencies on tangential wind from radial advection and total diffusion nearly balance, their radial gradients are also nearly equal and opposite. Though the resulting tendencies on the RMW are consistent with that predicted by K12, this is not for the reasons given by K12. The tangential momentum equation alone does not constrain the radial distributions of the tangential wind tendencies; rather, the distribution of radial velocity must also be known.

S15 further explored intensification and contraction from a dynamical perspective, using a linear vortex model [Three-Dimensional Vortex Perturbation Analysis and Simulation (3DVPAS)] to diagnose the separate contributions to the secondary circulation from diabatic heating and friction. We showed that in our control simulation, the direct effect of friction is to weaken the winds and to expand the RMW. The expansion tendency is *not* because the wind tendency is negative; rather, consistent with (1), it is because the radial gradient of the wind tendency is positive (see Fig. 13d of S15), as the peak frictional spindown occurs inward of the RMW. Using the WRF PBL tendency on tangential wind as forcing in 3DVPAS, we then showed that friction induces inflow, and this inflow yields a positive tendency on tangential winds. The positive tendency on tangential winds from friction-induced inflow is outweighed by the negative tendency from friction itself, so the net effect of friction is to spin down the wind field. However, the frictional inflow alters the spatial pattern of the tangential wind tendency such that the peak negative tendency on tangential winds is found outward of the RMW. Because the radial gradient of wind tendency at the RMW is then negative, the net effect of friction during the period examined is actually to contract the RMW. K12 argued that friction must act to oppose contraction, and while it is the case that the direct effect of friction is typically to expand the RMW, it is more meaningful to consider the combined effect of friction and the inflow induced by friction, and in S15, we showed that this combined effect can actually enhance contraction. This might explain why previous studies have found that the RMW contracts faster when surface drag is larger (e.g., Yau et al. 2004; Bryan 2013, KZ17). Even though both eyewall heating and surface friction resulted in a tendency to contract the RMW in our simulation, contraction did come to an end, and this was because of the rapid increase in the sharpness of the tangential wind maximum, rendering continued contraction more and more difficult.

As we discussed previously, the diagnostic equation of S15 merely represents a starting point for understanding contraction of the RMW. A complete dynamical explanation of contraction and the physical processes governing the cessation of contraction remains lacking. One way to move forward on this problem is to apply the diagnostic equation to the terms of the tangential momentum equation, that is, to substitute the tendency terms for $\partial V/\partial t$ into (1). In this manner, we could write (1) as

$$\frac{d\mathrm{RMW}}{dt} = \left.\frac{\dfrac{\partial}{\partial r}\left[u\dfrac{\partial v}{\partial r} + \dfrac{uv}{r} + w\dfrac{\partial v}{\partial z} + fu + \dfrac{C_D}{H}v\sqrt{(u^2+v^2)}\right]}{\dfrac{\partial^2 v}{\partial r^2}}\right|_{\mathrm{RMW}}; \tag{24}$$

here, we use the same notation as the tangential momentum equation [see (13)] introduced by KZ17. With the diagnostic equation in this form, it can be readily seen how it is the radial gradient of the tendency terms that influences the rate of contraction and not the tendency terms themselves. Although this still does not lead to a closed dynamical equation, it is possible to apply (24) to output from numerical simulations to gain further insight into the contraction process and the physical mechanisms that influence contraction.

6. Summary

S15 examined the relationship between contraction of the RMW and intensification of the maximum winds in TCs, and one aspect of that work was the derivation of a diagnostic equation for the rate of change of the RMW. Using both idealized simulations and observations, we showed that while there is commonly a rapid contraction of the RMW during the initial phase of intensification, the RMW often stops contracting long before peak intensity is reached. We used the diagnostic contraction equation to gain insight into this recently realized phenomenon. From a kinematic perspective, changes in the RMW with time are associated with two terms: the radial gradient of the time tendency of the tangential wind at the RMW and the curvature of the radial profile of tangential wind at the RMW. Although the first term is intuitive and widely recognized, the second term is often neglected, and there had been no previous study of the relative importance of the two terms. S15 showed that in their simulation, rapid contraction during initial intensification was due to the increasingly negative gradient of tangential wind tendency at the RMW, as might be expected based on previous studies. On the other hand, the rapid slowdown and cessation of contraction was not due to a decrease in this gradient, but instead it was due to the rapid sharpening of the profile of tangential wind at the RMW. Importantly, the curvature of the profile continued to increase throughout the simulation, rendering it more and more difficult for substantial movement of the RMW to occur. As it is typical for the radial profile of tangential winds to sharpen as TCs intensify (Willoughby et al. 1982; Willoughby 1990; Mallen et al. 2005), the results of S15 suggest that TCs may achieve their quasi-steady size as a result of this process.

In their comment on S15, KZ17 first argue that the analysis of S15 "provides little understanding of the processes underlying the RMW contraction." Although this judgment is ultimately up to the reader, we believe that S15 provided significant new insight into contraction of the RMW and its relationship with intensification, as summarized above. A specific criticism by KZ17 in this respect is that the contraction equation of S15 is merely diagnostic and kinematic and therefore does not yield dynamical understanding. Although S15 were clear about the limitations of their framework, diagnostic and/or kinematic approaches have a rich history of yielding advances in our understanding of TCs, and we believe that S15 adds to this knowledge as well.

Much of the comment of KZ17 concerns what they believe to be mistaken interpretations by S15 of the RMW contraction equations of K12 and W82. S15 stated that the equation of W82 was effectively a discrete approximation to the continuous and exact equation of S15, but KZ17 argue that the respective equations are not at all similar. In this reply, we have derived the equation of W82 from the equation of S15, demonstrating that S15 were indeed correct in their comparison.

In their comment, KZ17 argue that S15 were mistaken in their critique of K12 and assert that their equation is indeed correct and represents a valid dynamical framework for understanding the contraction of the RMW. In this reply, we have shown that the contraction equation of K12 and KZ17 is not correct. We also showed that from a conceptual and physical standpoint, the theory of K12 and KZ17 is inconsistent with the known kinematics of contraction. Specifically, any dynamical theory of contraction must depend in some manner on the radial structure of the wind field.

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